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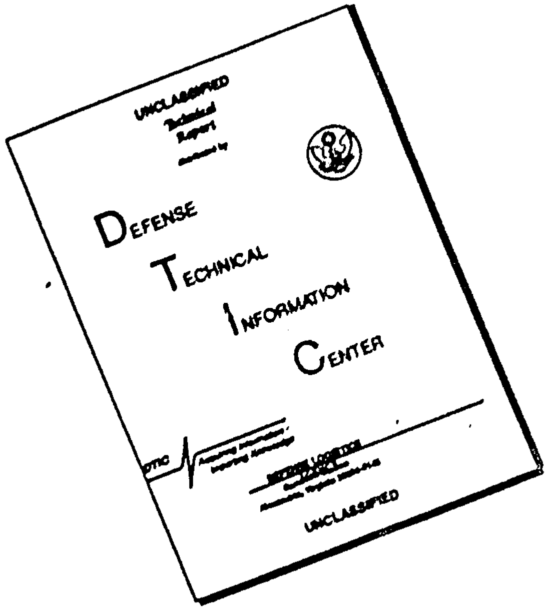

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FTD-TT-63-254

TRANSLATION

MECHANICS

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE
OHIO

NOV 4 1963

FTD-TT- 63-254/1+2

UNEDITED ROUGH DRAFT TRANSLATION

MECHANICS

English Pages: 296

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FTD-TT- 63-254/1+2

Date 28 August 19 63

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Moskovskoye Ordena Lenina i Ordena Trudovogo Krasnogo
Znameni Vyssheye Tekhnicheskoye Uchilishche im. Baumana

MEKHANIKA

Sbornik Statey

Pod redaktsiey
Dokt. Tekhn. Nauk Prof. S. S. Tikhmeneva

104

Gosudarstvennoye
Nauchno-Tekhnicheskoye Izdatel'stvo
Oborongiz
Moskva - 1961
Pages 3-181

FTD-TT-63-254/1+2

TABLE OF CONTENTS

The articles of this collection contain the results of scientific investigations in various fields of theoretical mechanics, analitical mechanics and their applications by the co-workers of the Department of Theoretical Mechanics of the Moscow Technical College imeni N. E. Bauman. Two of the articles pertain to the new theory of the calculus of variables and its application to the dynamics of rockets.

The book can be recommended to specialists in the field of basic theoretical mechanics, and also to the large circle of scientific workers and engineers who are working in the various fields of applied mechanics: dynamics of rockets, dynamics of flying machines, general applied gyroscopy, theory of automatic controls and the theory of vibrations.

INTRODUCTION

The present collection of articles written by the members of the Department of Theoretical Mechanics of the Moscow Technical College im. Bauman includes the results of scientific works that were performed at the Institute in the last two or three years. The majority of the authors are young scientific workers, but the results obtained by them are of definite theoretical and pratical interest, and, in a number of cases have an important scientific significance.

In connection with the fact that it is now 60 years from the day of birth and 35 years of scientific-pedagogic and social activity of the head of the Department of Theoretical Mechanics of the MTC im. Bauman, doctor of physical and mathematical sciences, professor Vladimir Vasil'yevich Dobronravov, the collection begins with an article dedicated to him, which has, substantially, been written by his students.

In the two articles pertaining to the mechanics of a system with non-holonomic coupling by V. V. Dobronravov which are published in this collection, there are indicated the perspectives of the future development of the methods of mechanics of non-holonomic systems; and some basic difficulties which arise in the connection with these methods are pointed out.

The third article written by V. V. Dobronravov, which follows closely a number of his previous works, once again demonstrates the fact that there exist relationships between composite quantities and their elements.

These so-called K-relationships are of theoretical interest, since they express the existance of complex dependencies; they also have a definite practical value. The article contains a number of K-relationships from various fields of mechanics.

A short remark by S. S. Tikhmenev gives a convincing explanation of one interesting phenomenon from the field of dynamics of a rigid body and gyroscopy. This phenomenon, which attracted the attention of the students of mechanics of the whole world, represents the self overturning of a gyroscope of an unique construction when put into motion upon a rough surface.

One's attention should also be called to the two articles by V. F. Krotov, which are a part of the cycle of his works, and which contain a new theory of the calculus of variables developed by him, which the theory includes the classical theory of Lagrange and Euler as a particular case.

The basic result which has been obtained by V. F. Krotov consists of the fact that the problem of the extremum of a function has, in addition to the solution on the curves of the equation of the so-called class C_1, along which the desired function that gives the extremum is continuous another solution in the class of broken curves. These solutions are expressed by discontinuous functions and they can be obtained without the necessity of integrating the differential equations of Lagrange-Euler. In conjunction with this it should be mentioned that the solutions of Krotov often give the absolute extremum in those cases when the problem has a Langrange-Euler solution.

The first of the two articles by V. F. Krotov contains some general assumptions which serves the purpose to make his theory more exact. The second articles demonstrates the fact that many problems of dynamics which are solvable by means of the calculus of variations require for their solution exactly the use of the V. F. Krotov method. So, for instance, it turns out totally unexpectedly that the optimal mode of propulsive force for a flying machine necessary for the achievement of a flight of maximum distance should be obtained by using a pulsing motor (non-continuous force.

The article by Yu. Ye. Zakharov and V. N. Baranov belongs to the field of automation and is a part of a cycle of serious investigations of hydromechanical processes[1] which take place in the slide valves of hydraulical mechanisms, which in the final analysis determine the effectiveness of the hydro-mechanisms action which were conducted by Yu. Ye. Zakharov. A number of the results of Yu. Ye. Zakharov's investigations were successfully applied by him to the design hydraulic vibrators for the automatic crushing of chips in the process of machining of articles on metal-cutting lathes. The articles also presents some of the new results of his investigation.

The first article by G. N. Knyazev, in which he examines in detail some problems of the theory of non-holonomic systems, which was the subject of lively discussions in scientific literature in the last ten years, is unique. In part, the results of the well-known Volterra equations, which employ the principle of total interchangeability of (the order of) integration and differentiation, and the modified results of equations of the given kind suggested by V. V. Dobronravov are analized in detail. It is shown that in the solution

of the Volterra type equations by the method of V. V. Dobronravov the principle of interchangeability is most correctly applicable, and the solution so obtained are applicable to the investigations of non-holonomic mechanical systems. The result that was obtained is verified by an actual application of the Volterra-Dobronravov equations to the formulation of the equations of one servo system-that of a stabilized gyro-frame. The stability criteria of this system as obtained from the equations formulated by this method of equations coincided with the result that has been obtained by academician A. Yu. Ishlinski, who used the methods that follow from the general theorems of mechanical systems to find a solution to the same problem.

In this manner G. N. Knyazev's article represents the first example of practical application of the methods of analytical dynamics of systems with non-holonomic couplings to the solution of concrete problems from the field of automation.

In his second article G. N. Knyazev has investigated the stability of the established motion of a loaded hydraulic motor with feedback, taking into consideration the compressibility of the oil and the deformation of the pipe-lines, using the Lyapunov method. On the basis of the subject of his investigations, the author makes recommendations, which, if followed, can result in the obtaining of stability of motion of a similar servo-motor.

The articles of V. A. Chikin contain interesting material relating to two new facts of the classical dynamics of a rigid body that were discovered during

the last decade. In 1948 the Italian scientist Grioli, using geometrical methods, has found a new interesting case of the motion of a rigid body containing a stationary point; the bringing to quadratures of the Euler equations for this case was performed by M. P. Gulyayev. A second Italian geometrician and student of mechanics, Signiorini, has established a new geometric concept, which he has called the "antiploe" of the straight line relative to an ellipse. The point which has been thusly denoted is related in a certain manner to the pole of the given straight line relative to the ellipse. It turned out that this point has a number of interesting geometrical and dynamical properties. The significance of the antipole concept to the field of dynamics of a rigid body has been discovered in a number of works of Signiorini and his students. The articles by V. A. Chikin contain explanations of new interesting properties of the antipole, deepening the understanding of this concept.

The articles by I. S. Chikina pertain to the field of dynamics of a rigid body having a liquid-filled cavity. Many of the problems in this field are of serious practical significance; as, for instance, the question of the influence of internal movement of fuel upon the flight (behavior) of a flying machine. In the same time, an exact solution of even simple problems requires the application of quite complicated mathematical methods.

In her first article, I. S. Chikina has described the usefulness of tensor and vector methods to the solution of problems of dynamics of a rigid body filled with a liquid.

The given methods give a clear and graphical description of the motions of the body and represent a more simple and convenient device for the discovery of new facts in the questions under study.

The second article presents the investigation of a concrete problem of the motion of a cylindrical cavity filled with a liquid. The problem is brought to a full solution and the results thus obtained may have significance in known practical applications.

N. K. Lobacheva examines in her article some problems of vibrations in clock-type mechanisms, using for this purpose the properties of delta-functions.

The article by K. A. Golenko shows that the question of stability of motion of an airplane taking into the consideration the effect of shifting of the liquid fuel in the (fuel) tanks reduces to the investigation of a system of ordinary equations.

The article by M. F. Khromenkov can be related to the dynamics of a rigid body as well as to the theory of vibrations. The author, whose works on the part played by friction in the motion of machine members and on the influence that friction has on wear have been frequently published, examines in his present work the problem of the influence of clearances in bearings and the eccentricity of masses upon the vibrations of motors. The methodology which is developed in the article can be successfully applied to the solution of problems of the influence of clearances and of the assimetricity of the distribution of masses in gyroscopic instruments upon the performance of the latter.

The article by V. P. Kutler is of the methodical kind; it can be useful in the teaching of theoretical mechanics and in student circles.

FOOTNOTES

Page 4. 1. One of the works of Yu. Ye. Zakharov has been published in the previous collection of transactions of the Department (Mekhanika, collection of articles, edited by V. V. Dobronravov, transactions of the MTC No 92, Oborongiz, 1959).

V. V. Dobronravov

SCIENTIFIC, PEDAGOGIC AND SOCIAL ACTIVITIES OF

By
V. V. Dobronravov

The year 1961 marked the sixtieth brithday and the end of 35 years of scientific, pedagogic and social activity of the head of the Department of Theoretical Mechanics of the MTC im. Bauman, doctor of physical and mathematical science, Professor Vladimir Vasil'yevich Dobronravov. During his years of service with the MTC V. V. Dobronravov has earned the respect of the staff and the love of students as a major scientist, demanding leader and sensitive educator.

Vladimir Vasil'yevich Dobronravov was born on March 17, 1901. On graduation from high school he entered the Saratov State University imeni N. G. Chernyshevski, graduating from its department of mathematics in 1924. His graduation thesis was in the field of theories of functions. As a result of his interest in the improvement of the teaching of mathematics and physics in high schools he has published an article on methods of teaching entitled "On the Teaching of Elements of Higher Mathematics in Secondary School."

In 1927, V. V. Dobronravov was invited to become assistant in the Physics Department of the Astrakhan' State Institute of Medicine imeni A. V. Luncharskiy. This was the beginning of his long time activity in the Soviet secondary school. During some period of time V. V. Dobronravov was interested in the problems of biophysics, and he has published a short article in the proceedings of the medical institute under the heading: "Neuro-fibrillate as a Physico-anatomical Substrate of Physical Processes." The basic idea of the article, which concludes that all sensory perceptions are received by the human brain

by means of vibrations of special fibers-neuro-fibriles, was due to his brother, N. V. Dobronravov, and the verification of the possibility for the existance of such a hypothesis by means of some elementary considerations from the field of the theory of vibrations was proposed by V. V. Dobronravov.

During the year 1929-1931, V. V. Dobronravov chaired the Department of Mathematics of the Saratov Agricultural Institute and was a reader of general mathematical courses in the Saratov State University.

In the beginning of the thirties, when the industrial might and scientific glory of our great Homeland was being born, there was a significant expansion of institutes of higher education in addition to the establishment of new schools; both endeavers requiring a large number of scientific workers and instructors. V. V. Dobronravov had the honor of working, in 1931, in a number of Moscow's higher educational institutions of world-wide fame: in the Moscow State University imeni M. V. Lomonosov, in the Moscow Technical College im. Bauman, in the Moscow Aviation Institute im. Ordzhonikidze and others. The most important part in his life was played by his lengthy work in the Moscow University with which he was associated from 1931 to 1952.

V. V. Dobronravov's interest in mechanics has increasingly become dominant. On studying the state and tendencies of the development of different fields of mechanics, he has noticed that the significance of analytical dynamics was underestimated by the Soviet students of mechanics in those years, and has felt the necessity of scientific investigation in this field. On the other hand, he has witnessed the development of non-holonomic geometry, which represents the

geometrical expression of the properties of mechanical systems with non-holonomic couplings.

The motion of a system with non-holonomic couplings can be considered as the motion in the non-holonomic multiformity whose measurement is dependent on a numbers of non-holonomic couplings. In this manner all the methods of the tensor multidimensional differential geometry and the entire theory of non-holonomic multiformities can be successfully employed in the examination of non-holonomic systems.

A large role in the final choice of the direction of his scientific work was played by V. V. Dobronravov's participation in a seminar on tensor methods and their applications which was organized by professor V. F. Kagan. The head of the Department of Theoretical Mechanics of the MSU, professor B. N. Bukhgol'ts has fully approved V. V. Dobronravov's intention to take as a serious interest in the mechanics of non-holonomic systems and has invited V. V. Dobronravov to become a member of the department.

In his works on the mechanics of non-holonomic systems V. V. Dobronravov has attempted to fill the void between the work of Russian scientists in the beginning of this century and the later work of foreign scientists that has been created in this field. Thusly, he has shown that the equations of S. A. Chaplygin and I. Tsenov are self-transformable. In the course of this, advantages of the Tsenov equations over those of Appel became apparent. He has also derived the equations of motion of systems with linear non-holonomic couplings of the

Volterra type, but in generalized Lagrangian coordinates. These equations were applied to the solution of the Karateodori problem of the motion of sleds.

In the same time V. V. Dobronravov has examined the problem of non-linear non-holonomic couplings. A basic difficulty became apparent in the process of obtaining solutions of equations of motion with similar couplings since the presence of these couplings has made it impossible to use the principle of D'Alembert-Lagrange due to the absence of linear relatively permissible relationships. For non-linear non-holonomic couplings of the form

$$f(x_\nu, \dot{x}_\nu, t) = 0 \qquad (1)$$

N. G. Chetayev has suggested to employ the following relationships in the solution of equations of motion

$$\sum \frac{\partial f}{\partial \dot{x}_\nu} \delta x_\nu = 0, \qquad (2)$$

which have with time acquired the name "Chetayev's relationships." V. V. Dobronravov was the first to attempt to prove the relationsips (2) formally, applying Taylor's series to the function(1).

The above mentioned works of V. V. Dobronravov have been published in his lengthy article "On the Equation of Motion of Non-holonomic Mechanical Systems with Linear and Non-linear Couplings" (Proceedings of the Moscow Hydro-Metereological Institute, first printing 1939, pp 273-215).

In 1938, on the recommendation of hhe Moscow Hydro-Metereological Institute V. V. Dobronravov was awarded the academic title of docent and in May of 1939, on

the basis of the results of his works, he has defended a dissertation for the obtaining of the title of candidate of physical and mathematical sciences of the MSU. His opponents were Corresponding Members of the Academy of Sciences USSR L. N. Sretenski and Professor B. V. Bulgakov.

In September 1942, V. V. Dobronravov was called to the position of docent of the Department of Higher Mathematics of the Moscow Aeronautical Institute imeni Sergo Ordzhonikidze. In the mean time he has continued his work in the capacity of docent of the Department of Theoretical Mathematics of the MSU, and in 1945 he was competitively chosen as the official docent of the department.

The development of the theory of non-holonomic couplings has created a more sharp division of classical analytical mechanics into the mechanics of non-holonomic and holonomic systems. Their content turned out to be profoundly different. The mechanics of holonomic systems had its beginnings in the classical Lagrange equations of the second type, possessing a number of remarkable properties, owing to which this class of different equations has acquired tremendous significance not only in mechanics but also in the development of a number of fields of mathematical systems and of theoretical physics. However the general Lagrangian equation are unfit for the investigation of mechanical systems with non-holonomic couplings. There has arisen the problem of finding of the most simple and substantial generalization of all basic concepts of classical mechanics for the case of non-holonomic systems. It was obvious that this generalization will become possible only in the case that first there will

be found equations of motion, which will, first, be applicable to holonomic as well as to non-holonomic systems, and, second, they will have a totally identical structure. In order to obtain these equations it was necessary to pass from the generalized Lagrangian coordinates to some other (coordinates) and to write the required equations in the latter.

In analyzing different methods of formulating the equations of motions in the presence of non-holonomic couplings, V. V. Dobronravov noticed that the sought-after transformation already exists. It was first introduced into mechanics by V. Volterra, and then L. Boltzman and Hamel gave it a form which is very convenient for the obtaining of solutions of the equations of motion. This transformation is not entirely simple; it is expressed not by final relationships between initial Lagrangian coordinates and the new coordinates, but by equations which relate the differential (forms) of old and new corrdinates, as follows:

$$d\pi^{\rho} = \sum_{\mu=1}^{n} a^{\rho}_{\mu}\, dq^{\mu}. \qquad (3)$$

The differential of the new coordinates actually exist and they can have physical meanings, but the coordinates proper, in their final form, are not given. Volterra called the time derivatives of these coordinates the "kinematic characteristics" of the mechanical systems and derived from them his equations of motion.

In such a manner, the Volterra equations turned out to be applicable to both holonomic and non-holonomic systems and in the mean time were of entirely identical structure in both cases. Boltzman and Hamel and the analytical geometricians have provided the Volterra transformations with a convenient "differential" symbolism, by introducing the most successful term "non-holonomic

coordinates" (quasi-coordinates). An exact system for the operation with non-holonomic coordinates was created.

V. V. Dobronravov has performed the important service of introducing the given method into dynamics. He has demonstrated the significance of non-holonomic coordinates not only for systems with non-holonomic couplings, but also for holonomic systems. It has turned out that in many cases the equations of motions of even holonomic systems can be written more expediently in non-holonomic coordinates. As an example of the latter we can use the classical problem of the motion of a rigid body around a stationary point. The dynamic Euler equations are in fact transformed into non-holonomic coordinates.

The properties that have in this case been derived from non-holonomic coordinates are the projections of instantaneous angular velocities onto the moving axes of the coordinate system. In this same case the nature of non-holonomic coordinates given by differentials can be easily elucidated; these differentials of the motion of a rigid body around a stationary point represent infinitesimally small angles of revolution of the body around respective instantaneous axes.

In a number of articles, which were mainly published in the Reports of the Academy of Sciences USSR[1] during the forties, and in more detail in his doctoral dissertation "Analytical Dynamics in Non-holonomic coordinates," V. V. Dobronravov has shown that the entire dynamics of holonomic systems will look like in non-holonomic coordinates. He has proven the theorems of Poisson and Laurent in non-

holonomic coordinates, as well as the generalized Hamilton-Jacoby theory for the case of non-holonomic coordinates - the theory of the residual multiplier; the theory of integral invariants and Lagrange's theory of frames. Applying the device of non-holonomic coordinates to the mechanics of holonomic systems, he has obtained a new and important result - the integral invariant of the sixth order for the motion of a rigid body around a stationary point. In one of his works (Herald of the MSU, 1959, No 9) V. V. Dobronravov has applied non-holonomic coordinates to the mechanics of continuous media (partially to hydrodynamics). With the help of this method he has succeeded in exposing some new facts of the theory of spiral motion of fluids.

If the non-holonomic coordinates are properly chosen (when the left-land members of the equations of the couplings are acceptable as the differentials of certain non-holonomic coordinates), then the equations of motion of systems with non-holonomic couplings have exactly the same structure as the equations of motion of holonomic systems in non-holonomic coordinates, but the number of equations of motion in the case of non-holonomity of the couplings are lessened by the number of couplings. The last condition does not permit the automatic transfer of all theorems and assumptions of the dynamics of holonomic systems into the case of a system with non-holonomic couplings.

V. V. Dobronravov has shown that it is possible to generalize the Hamilton-Jacoby theorem and to obtain integral invariants for those non-holonomic systems which were examined by S. A. Chaplygin and were called "closed non-holonomic systems"

by V. V. Dobronravov. Such a term was introduced on the basis of the fact that the equations of motion that were set up for similar systems by the method of non-holonomic coordinates, when the coordinates were properly selected, represent by themselves the equations of motion in a certain closed sub-space of one independent system of coordinates. Although only time-independent coordinates can be found from this equations, it is further possible to determine time-dependent coordinates from the equations of non-holonomic compendencies. These systems can be called "systems with a full number of cyclical coordinates." V. V. Dobronravov has shown that beginning with the equation of motion of closed non-holonomic systems, set up in non-holonomic coordinates, it is possible to obtain the generalization of classical theorems of analytical dynamics for such non-holonomic systems. In the works of V. V. Dobronravov there are brought examples of application of the Jacoby method to the problem of rolling of a homogenous ball on a flat surface and to one particular case of the Karateodori problem.

V. V. Dobronravov's investigations in the field of dynamics of non-holonomic systems were presented in his dissertation submitted for the purpose of obtaining the academic rank of doctor of physical and mathematical sciences, which were defended on June 25, 1946 in the Scientific Council of the Department of Mathematical Mechanics of the Moscow State University. His opponents were major Soviet scientists: corresponding members of the Academy of Science USSR N. G. Chetayev and professor V. V. Vagner and Bulgakov. Favorable reactions to the dissertation were presented by professor G. N. Sveshnikov and A. A. Vlasov.

In 1947 V. V. Dobronravov, as one of the leading scientists in the field of analytical dynamics, was entrusted with the writing of an articles about the development of analytical dynamics in USSR for the collection "Mechanics in the USSR during 30 Years."

From 1946 to 1951, V. V. Dobronravov headed the Department of Theoretical Mechanics of the Moscow Institute of Chemical Machinery Building, where he was competitively selected and certified by the Higher Attestation Committee for the academic rank of professor.

From 1951 V. V. Dobronravov heads the Department of Theoretical Mechanics in one of the oldest Moscow Higher educational institutions - in the Moscow Technical Colleges im. N. E. Bauman, the chair that was in this time occupied by N. Ye. Zhukovskiy, A. P. Kotel'nikov, V. P. Vetchinkin and others.

The Moscow Educational College has in this time gradually entered into a new phase of development. Together with general machine design the school began developing instrument design, new departments connected with newly developed technology were created and the admission of students increased. In order to assimilate modern technology, preparation in such special fields of mechanics as the stability of motion, gyroscopy, vibrations, variational method etc was necessary.

Under the leadership of V. V. Dobronravov the staff of the department of theoretical mechanics has successfully solved probelms of the methodology of giving a course in theoretical mechanics in an eduactional College of a wide cross section. It was decided that

only those topics which are absolutely necessary for the student's further study of specialized topics and above mentioned divisions of mechanics should be left in the programs of lectures and exercises, and that the vector method that not only provides for great saving in time during lectures and reports, but also have a scientific and perceptual values of their own. The department co-workers began to read required as well as elective courses on the theory of vibrations, variational methods, ballistics and others in a number of faculties. Steps were taken for strict unification of the process of studies by all the faculties and groups.

A special mention should be made about the development of teaching equipment for the demonstration rooms of theoretical mechanics which was brought about by the department. A program of laboratory experiments in theoretical mechanics was also put together. Thusly the methodological work of the department has in this manner passes beyond the confines of the MTC. V. V. Dobronravov himself is one of the most active members of the methodological commission on theoretical mechanics at the Ministry of Higher and Intermediate Special Education USSR. This commission has done a lot of work in order to work out a program of theoretical mechanics for higher educational institutions taking into consideration the problems of the new technology.

Serious difficulties were encountered in the organization of the scientific works of departments. It would seem reasonable to assume that the general-scientific departments should concern themselves with the problems of the further development of the disciplines that they represent, and they should also engage in the development of new theoretical aspects. On the other hand,

the traditions of one of the oldest technical higher educational institution have demanded active participation by the general-scientific departments in the solutions of practical problems, which are related to the technical progress of our industry.

The organization of scientific work of a department is closely related to the training of specialists and to the increase in their number, something that is mainly accomplished by the institution of fellowship. However the training of fellows in theoretical mechanics that is carried out in a technical College is connected with certain difficulties, in part due to the fact that it is difficult for fellows with a university education to perform the experimental part of a dissertation having a technical character. For this reason fellows in theoretical mechanics are lacking even in the major higher education institutions.

Notwithstanding these difficulties, a resolution favoring the establishment of fellowships at the department of theoretical mechanics of the MTC in Bauman was adopted on the initiative of V. V. Dobronravov. Into fellowship were accepted persons with university education, who had previously worked in the field of applied mechanics, or who had an engineering education in specialties which were related to general and analytical mechanics.

The lack of own educational facilities at the department and the variety of dissertation topics has significantly contributed to the difficulty of managing the fellowship program. It was not so easy for V. V. Dobronravov himself to choose the topics of dissertations, and to establish connections with plants and other organizations.

The improvement in the setting up of the fellowship program was extremely helpful in organizing of a student scientific circle under the leadership of V. V. Dobronravov and in the establishment of the practice of readings in special divisions of mechanics in the more advanced courses in the department. As a result of these measures it was possible to attract to the fellowship program a special kind of young people, from whose ranks there were trained a number of highly qualified technical workers. The majority of dissertations represented by themselves serious work, which contained original and important results, for instance suudies of the precision of gyroscopic instruments, on the theory of vibrations, theory of automatic controls, in the fields of hydro-mechanical processes etc.

The works of Fellow V. F. Krotov, who has obtained new and important results in the field of the calculus of variations and who successfully applied them to the contemporary problems of applied mechanics deserves a special mention.

The virtue of V. V. Dobronravov's work consists in the fact that e has lead the researchers of a young scientist into a little investigated field of knowledge, as a result of which a new chair that of the calculus of variations was created.

The training of scientists has permitted begin the process of a wider expansion of scientific investigational work at the department. In order to increase the number of topics that are taught and to con-

centrate the efforts, a resolution about the development of two basic directions in the scientific work was adopted: a) on the general and applied gyroscopy and automation and b) on the dynamics of flight, taking into consideration that both of these directions have common points as for instance in the field of flight control. Thusly a group of the department has done major work in the field of investigation of the precision of a gyrosystem taking into consideration the elasticity of coupling of the system's elements and its clearances. At present the department conducts investigations of hydraulic servomechanisms.

Busy as he is with the leadership of a general-scientific department, and with the direction of a fellowship program with varied fields of interest, V. V. Dobronravov does not cease his occupation with the mechanics of non-holonomic systems. V. V. Dobronravov's work attract the attention of scientists in the Soviet Union as well as abroad.

Due to the character of the relationships imposed on non-holonomic systems some problems came up that required additional development and refinement of the theory of these systems. The problem that should primarily be mentioned, is the one that has subsequently acquired the name of the problem of interchangeability of the order of differentiation and integration in respect to time. In the solution of the equations of motion in non-holonomic coordinates it was necessary to employ the so-called "Beltrami relationship" and to operate with expressions of bili-near co-variants of the coordinates of the mechanical system. In the transformation from the Cartesian coordinates of the system to its non-holonomic coordinates, Volterra has set the bilinear co-variant of all Cartesian coordinates equal to zero. This assumption could be justified

from the point of view of non-holonomic multiformities, developed by Vrancheanu, according to which a closed loop, in which the operation $d\delta - \delta d$ is performed, is transformable into an open loop only in the space defined by non-holonomic coordinates. The well known Soviet student of mechanics, G. K. Suslov has proposed that in general the bilinear co-variant should be made equal to zero; that is, the interchangeability of differentiation and integration should be employed only for independent non-holonomic coordinates of a system. The analysis of this problem has turned into a discussion during which the methodology of the solution of Volterra equations was touched upon. V. V. Dobronravov took part in the discussion and has published two articles; in the magazine "Prikladnaya Matematika i Mekhanika" (Applied Mathematics and Mechanics) and in the proceedings of the department (Oborongiz, 1956).

Another question which was not sufficiently clear was that of the exclusion of dependent accelerations and velocities from the expressions for the kinetic energy in the case of systems with non-holonomic relationships. Appel and Volterra transform these expressions, eliminating the dependent velocities and accelerations from them before the differentiation in respect to independent velocities and accelerations, and Gamel performs these transformations after differentiation. The equations of S. A. Chaplygin are of intermediate character - they contain expressions for kinetic energy in both the initial and transformed form. V. V. Dobronravov has suggested a method of transformation to such non-holonomic coordinates for which the kinetic energy takes tbe normal quadratic form only from independent generalized non-holonomic velocities. Then the results of differentiation will be identical in all cases.

In his last works, which were read at the Fifth All-Union Conference on theoretical and applied mechanics, V. V. Dobronravov has described those inconsistencies which can be discovered in the scientific literature in the question of the definition of the concept of virtual translations for systems with non-linear non-holonomic relationships. At the Fourth All-Union Mathematical Conference in July of 1961, he has presented a paper on the problem of the utilization of the first degree integrals of dynamic equations in the capacity of non-holonomic relationships.

During his occupation with the problem of the motion of a rigid body around a stationary point, V. V. Dobronravov has developed the relationships between the kinetic energy of a rigid body, the modulus of its kinetic moment, the body's angular velocity, its components and the projection of its kinetic moment on the vertical (see Transactions of Moscow Institute of Chemical Machinery No 2 (10), 1950). These relationships are an example between composite quantities and their components. A number of relationships of this kind from various fields of mechanics are contained in the presence of collection. A particular case of the above relationships for the Euler case and for the condition of dynamics symmetry of the body, has permitted to establish a curious fact which consists in that the unitary vector of kinetic moment described a circumference on the gyrational ellipsoid of a rigid body (see Reports of the Academy of Sciences USSR, 1949 Vol. LXV, No 2)[2]. From this very same relationship it is possible to obtain a certain limiting relationship for the Lagrangian case of angular velocity of nutation of a rigid body. A lengthy article on the Euler case was published by V. V. Dobronravov in the Scientific Notes of the MSU in 1952.

In the course of his work in the Moscow Technical College V. V. Dobronravov has increasingly more interested hi self with the application of theoretical mechanics and mathematical methods to technology. He has published an article in the magazine "Avtomatika i Telemekhanika" (Automation and Telemechanics) (Vol 17, No 3, 1956), in which he sets forth a quite simple sufficient criterion for the stability of linear systems of differential eqautions with constant coefficients; the given criterion is based on Adams' theory of determinants. An article concerning the application of the theory of non-holonomic systems to the problems of automation is published in the present collection.

V. V. Dobronravov has accumulated a lot of experience in the field of methodology of teaching of theoretical mechanics and of a number of mathematical subjects, which is mirrored in the teaching aids composed by him: the manual "Mekhanika" for the students enrolled in the correspondence courses of the Moscow Hydro-Metereological Institute methodical guidance of the course of mechanics for the correspondence divisions of the department of mathematical mechanics and the department of physics of the MSU, a manual for the lecture course "Differential Equations of Mathmetical Physics" for the fellows of the Moscow Aeronautical Institute. In the (static, kinematics, dynamics) were created by the entire staff of the department under the editorship of V. V Dobronravov. The department's work of creation of study manuals is continuing The publication of collections of works by the members of the department has been organized.

V. V. Dobronravov has set for himself the task of creating a mo-

nograph on non-holonomic systems for the purpose of generalization of his labor of many years in this field, and also to set up an original course of theoretical mechanics taking into consideration the specific problem of MTC. It should be hoped that, despite his preoccupation with pedagogic scientific and social work, Vladimir Vasil'yevich will be able to complete this task.

V. V. Dobronravov is a scientists with many sided interests. He is an actual member of the Moscow Mathematical Society (from 1944 on) and a member also of the Moscow Society of Investigators of Nature. In 1952, at a joint meeting of these two societies, Vladimir Vasil'yevich has read a paper about the scientific activity of academician S. A. Chaplygin in connection with the tenth anniversity of the latter's death.

In the beginning of the fifties the attention of V. V. Dobronravov was attracted by astronautics. He has kept one eye on the problem of interplanetary communications even before, although these problems were very far removed from the attention of the wide circles of society.

In 1951, V. V. Dobronravov has organized at the department a student's astronautical circle which included students attending junior courses. As a result of this popular scientific lectures were heard alongside with scientific lectures that required the application of a strict mathematical and mechanical analysis. Many members of the circle have with some time become scientific workers in the fields of science allied with astronautics. Some of the works of the circle's members were given prizes by the Ministry of Higher Education.

In 1954 an astronautical section was organized at the Central Aeronautical Club imeni V. P. Chkalov. This section conducts useful activities for the popularization of the basic problems of astronautics. V. V. Dobronravov is taking an active part in the section's work in his capacity of scientific supervisor of the section.

In 1954, in the New Year's edition of the newspaper "Komsomolskaya Pravda" there appeared an article by V. V. Dobronravov entitled "In a Moon Rocket." Such an article with its elements of science fiction was needed at that time, and especially in a newspaper devoted to youth. During that year the "Znaniye" (Knowledge) Publishing House has published a popular scientific pamphlet written by V. V. Dobronravov entitled "Kosmicheskaya Navigatsiya" (Cosmic Navigation), which was the first publication of its kind to come out in the post-war years. In this pamphlet, the author, in simple form, but on the basis of strict percepts of celestial mechanics explains in what manner will the flights of space ships in the boundaries of the solar system be performed. As in all his other works on popular astronautics that were published in a large number of newspaper and magazine articles, V. V. Dobronravov has attempted to bring clarity to the understanding of those factors upon which the trajectory of the cosmic flying machine and the law of its motion depends in the absence of the propelling force. In a number of articles published in the journal "Kryl'ya Rodiny" (Wings of the Homeland) V. V. Dobronravov has explained from the scientific point of view also such phenomena as overloading which is experienced by the crew of a cosmic ship at the ship's accelerations or decelerations and the state of weightlessness during the motion of a satellitein a closed orbit.

And then the day that has entered in to the history of mankind has arrived - October 4, 1957, when the USSR has launched the world's first artifical earth satellite. Articles by Vladimir Vasil'yevich, who never refuses to take part in the blessed work of popularization of the triumphs of Soviet science and technology have appeared in many of the capitol's newspapers and magazines. More than once has he appeared on radio and television, and he has written many articles for the foreign press. In this periods V. V. Dobronravov's qualities as a scientist, lecturer, popularizer and commentator have been especially displayed.

In the spring of 1958, he has received an invitation from the Society for Soviet-German Friendship to become a member of the USSR delegation that was taking part in the celebrations of the German-Soviet Friendship Day and he has appeared at the session of the German Academy of Sciences with the story of the successes of the Soviet scientists in the investigations of the universe.

V. V. Dobronravov even now does not give up his activity in the field of astronautics. During the performance of further brilliant experiments for the conquest of outer space in the Soviet Union, such as the launching of Soviet cosmic rockets towards the moon, the taking of pictures of the other side of the moon, the testing of mighty ballistic rockets which have descended into the Pacific Ocean with great accuracy, the launching into orbit of heavy earth satellite ships, the return to Earth of ships with living beings, and finally, the historic flights into space by soviet fliers-cosmonauts Yu. A. Gagarin and G. S. Titov, V. V. Dobronravov has always appeared with commentaries in the press, on radio, television and in motion picture magazines, endeavoring to widen

the horizons of his reading and listening audience by various remarks of technological and scientific nature and by considerations pertaining to the further investigations of outer space. So, for instance, in his articles on the first Soviet satellite V. V. Dobronravov has touched upon the problem of the vacillations of space ships about their own center of gravity. In other articles he has suggested one of the possible schemes for the returning of a space ship to Earth.

V. V. Dobronravov has appeared with articles on other problems also. So, for instance, in a lengthy articles published in the newspaper "Literatura i Zhizn" (Literature and Life) of June 15, 1961 he has commented on the All-Union Conference of Scientific Workers.

The Soviet Government has highly prized the pedagogic and social-scientific activities of V. V. Dobronravov which it expressed by awarding him the worker's Red Flag and other medals.

For his scientific and popularizing work V. V. Dobronravov has received a certificate of merit from the Student of Scientific and Technological Society im. N. Ye. Zhukovski at the MTC a certificate of merit from the magazine "Kryl'ya Rodiny", a certificate of merit from the Moscow Municipal Section for the Dissemination of Political and Scientific Sciences etc.

At the present time V. V. Dobronravov is in the midst of the following of his creative powers; guiding the work of an important general-scientific department of the MTC and continuing the honored work of education of youth. We are certain that Vladimir Vasil'yevich Dobronravov

will for many more years work productively for the benefit of our Homeland.

S.S.Tukhmenev,	M.P.Gulayev,	V.K.Bocharov,
V.P.Tronina,	Yu.Ye.Zakharov,	Ye.K.Shigin,
V.A.Chikin,	I.S.Chikina,	V.F.Krotov.
G.N.Knyazev,	V.I.Lyamin,	

FOOTNOTES

1) Page 17. According to the presentation of academician A. N. Kolmogorov.

2) Page 26. According to the presentation of academician A. I. Nekrasov.

THE PRINCIPLES OF D'ALEMBERT-LAGRANGE AND HOLDER AND THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS WITH NON-HOLONOMIC COUPLINGS[1]

by

V. V. Dobronravov

In the basis of analytic dynamics there lies two basic concepts: the concept of relationship and the concept of permissible transposition. However, the properties of these very relationships - holonomism or non-holonomism, dependence or independence in respect to time, idealness or non-idealness, one-sideness or two-sidedness- have a strong influence on the application of these concepts to the solution of equations of motion.[2] Each one of the above properties may become apparent depending on its development and mode of expression. So, for instance, if at the inception of the mechanics of nonholonomic systems only systems with non-holonomic relationships of the first order were really known, then in the development of present day technology, when other kinds of physical influences different from those examined in classical mechanics act on mechanical systems, it becomes necessary to study non-linear and higher degree non-holonomic relationships.

The same can be said about rheonomic relationships. If up to now it was customary to consider motion of systems with rheonomic relationships as a motion in a certain deformable space (A. Wundheiler), then in the present time with the development of the mechanics or controlled motion, rheonomic relationships can be regarded as some program to which the motion of the system must conform and whose execution should be assured by actions of a certain kind partially by the reactions of relationships although in a wide sense of this concept.

The equations of motion of the material points system, regardless of the nature of the relationships, can be derived by getting away from the unified D'Alembert-Lagrange principle, the applicability of which the mechanical system is beyond any doubt, because it is evident from the formulation: "the sum of the elementary function of all forces directly applied to the points of the system and D'alembert inertial forces should equal zero at any possible displacement of the system." The above assumption, as well known, is expressed by the general equation of dynamics

$$\sum_{\nu=1}^{N}(-m_\nu \bar{a}_\nu + \bar{F}_\nu + \bar{R}_\nu)\,\delta\bar{r}_\nu = 0, \qquad (1) \qquad (1)$$

where a is the acceleration;

F and R are the active forces and the reactions of the relationships.

In this case under the term permissible translation is understood a more simple concept which has been introduced by Lagrange; a translation which is at the given moment permitted by the relationships imposed upon the system. However, the concept of permissible translation requires at the present time additional examination and exactitude, which will quarantee that it will be possible to obtain from the general equation of dynamics (1) reliable equations of motion for certain mechanical systems. Undoubtedly/ it can be considered as correct for holonomic time independent relationships that is, for relationships which are expressible by final equations in respect to the coordinates of the system.

$$f_j(x_\nu) = 0, \quad (\nu = 1, 2, \ldots 3N),$$

where 3N is the number of all Cartesian coordinates of the system's par-

ticles.

In this case the permissible transformation satisfy the conditions

$$\sum_{\nu=1}^{N} \frac{\partial f_j}{\partial x_\nu} \delta x_\nu = 0.$$

Undoubtedly this concept is valid for the case of linear non-holonomic relationships which are time independent, that is for relationships having the form

$$\sum_{\nu=1}^{3N} b_{s\nu} dx_\nu = 0;$$

in this case the permissible transformations must satisfy the relationships.

$$\sum_{\nu=1}^{3N} b_{s\nu} \delta x_\nu = 0.$$

However, already after the discovery of linear non-holonomic relationships towards the end of the 19th century the concept of permissible transformation of a system had to be more exact in the case of rheonomic relationships. In the course of solving the equations of motion of mechanical systems with linear non-holonomic relationships it was begun to associate still another assumption to the principle of Lagrange-D'Alembert, which could be called "Holder's principle" since it was the latter who has most clearly formulated the following postulate in one of his works (in 1896) in the presence of any linear relation of the form

$$\sum_{\nu=1}^{N} \bar{b}_{j\nu} d\bar{r}_\nu + b_j dt = 0 \quad (j = 1, 2, \ldots, l) \qquad (2)$$

the permissible transformations of the system must satisfy the rela-

tion[3].

$$\sum_{\nu=1}^{N} \bar{b}_{i\nu} \delta \bar{r}_\nu = 0 \qquad (3)$$

In this case the vectors $b_{i\nu}$ and scalars b_i are only dependent on r and the time t. In this manner when rheonomic relationships are present in the system a sharp differentiation between the space of permissible transformations ($\delta \bar{r}_\nu$), and the space of actually transformations ($d\bar{r}_\nu, dt$) appears.

Although the above situation did attract the attention of a number of authors (Kirckhoff, Wundheiler, Vranceanu, Gugino, Quarleri and others) still its significance in the investigation of systems with rheonomic relationships has not yet fully been explained.

The main source of permissbble properties for the case of application of the D'Alembert-Lagrange principle in the solution of equations of motion of systems with relationships (2) is based on the fact that the equations of actual motion of the system are solved as equations of motion in a space of permissible transofrmations, although the latter are subject to parametrical changed (with respect to time).

However, a number of relationships which are satisfiable in one space have no place in another. For instance, the relationships (2) may not be integrable, while the relationships (3) may be integrable. This fact alone may lead to the non-correspondence of the motions in both spaces. Furthermore, relationships can be ideal in one space, and non-ideal in the other; since the relationships that give the ideality of relationships that is equations

$$\sum_{\nu=1}^{N} \overline{R}_\nu \delta \overline{r}_\nu = 0, \qquad (4)$$

which are satisfiable, for instance, for the conditions (3), do not necessarily indicate that similar relationships in the space $(d\bar{r}_\nu, dt)$.
The above circumstances is especially important, since as a result of it, it may turn out that in the actual motion the system of equations which describe the motion, will be much wider in the sense of possibility of finding of both the motions and the reactions of the relationships. A system with a rheonomic relationship may turn out to be dynamically indetermined.

In regard to the technical applications of such systems, the latter may quite possibly create no difficulties, and what is most probable, it may in the opposite, cause no propose the problem of optimal choice, from this or that point of view, of a motion, by way of assignment of proper reactions $\overline{R}_V$ of rheonomic relationships.

A motion which can be found for the conditions (4) is one of the variants of motions. However, in order for it to correspond to actual motion it is necessary to give some additional conditions for relationships (2). This follows from the form of the reactions of relationships, which are found for the conditions (4).

At the present time the methods of analytical dynamics begin to be widely applied to the theory of controlled and regulated systems. Here the finding of the reactions between relationships acquires major significance; it is possible to consider a relationship to the definition of some of the program of motion,

and the reactions between relationships as causes that determine the program. The classical dynamics in the whole has mainly investigated systems with ideal relationships, for which many types of equations of motion of mechanical systems with ideal non-holonomic relationships were obtained, among then also equations with undetermined multipliers.

The remaining types of equations are distinguished by the large variety of their inner form and by the methods of their solution which sometimes cause variances in a number of questions.

If one were to introduce reactions of non-ideal relationships into certain types of equations, then one may obtain equations which will cause unnecessary complications in the computations of reactions of relationships.

Some types of equations, even for ideal relationships, can be only solved subject to substantial limitations. So, for instance, the S. A. Chaplgin equations are applicable only to systems with ideal relationships which are independent of time, and which have cyclical coordinates.

In the present remark there are given the solutions to the most general equations of motion for all categories of non-holonomic relationships, including the non-ideal, time dependent and those that do not have cyclical coordinates. Moreover, should the necessity arise to calculate the reactions between relationships, it will be done for the most simple cases. The solutions of equations for not require the application of variation principles that are related to the concept of

permissibility of transformation, which assures the uniqueness of the equations.

Let us assume that we are examining a system of particles, whose geometrical configurations, taking into consideration holonomic (rheonomic) relationships are defined by general Lagrangian coordinates q1, q2....qu. This means that all the Cartesian coordinates of the particles can be expressed by q_i and the time t in the following manner:

$$x_\nu = x_\nu(q_1, \ldots q_u, t);\ y_\nu = y_\nu(q_1, \ldots q_u, t);\ z = z_\nu(q_1, \ldots q_u, t) \qquad (5)$$

The velocities of the particles will be expressed by means of generalized velocities:

$$\left.\begin{aligned} \dot{x}_\nu &= \sum_{j=1}^{u} \frac{\partial x_\nu}{\partial q_j} \dot{q}_j + \frac{\partial x_\nu}{\partial t}; \\ \dot{y}_\nu &= \sum_{j=1}^{u} \frac{\partial y_\nu}{\partial q_j} \dot{q}_j + \frac{\partial y_\nu}{\partial t}; \\ \dot{z}_\nu &= \sum_{j=1}^{u} \frac{\partial z_\nu}{\partial q_j} \dot{q}_j + \frac{\partial z_\nu}{\partial t}. \end{aligned}\right\} \qquad (6)$$

We will further assume that in addition the following linear non-holonomic, rheonomic; and generally speaking also non-ideal relationships are imposed upon the system

$$\sum_{j=1}^{u} A_{sj} \dot{q}_j + A_s = 0, \quad (s = 1, 2, \ldots l) \qquad (7)$$

where A_{sj} and A_s are functions of q_j and t.

If we express l generalized velocities from (7) by means of the remaining n=u-l generalized velocities and substitute into (6) we will obtain

$$\dot{x}_\nu = \sum_{i=1}^{n} b_{\nu i}\dot{q}_i + b_\nu \qquad (\nu = 1, 2, \ldots 3N), \tag{8}$$

where x_ν are Cartesian coordinates.

We will write the Newtonian dynamic equations of all the particles

$$m_\nu \ddot{x}_\nu = F_{\nu x} + R_{\nu x} \quad (\nu = 1, 2, \ldots 3N), \tag{9}$$

where $F_{\nu x}$ are the components of active forces and $R_{\nu x}$ are the components of the forces of reactions between relationships.

Multiplying each of the equations (9) by $b_{\nu i}$ and summing on ν for each i, we will obtain

$$\sum_\nu m_\nu \ddot{x}_\nu b_{\nu i} = \sum_\nu F_{\nu x} b_{\nu i} + \sum_\nu R_{\nu x} b_{\nu i}; \quad (i = 1, 2, \ldots n) \tag{10}$$

Furthermore, we have

$$m_\nu \ddot{x}_\nu b_{\nu i} = m_\nu \frac{d}{dt}(\dot{x}_\nu b_{\nu i}) - m_\nu \dot{x}_\nu \frac{db_{\nu i}}{dt}. \tag{11}$$

It is evident that on the basis of (8) there exists the following wing classical Lagrange relation:

$$b_{\nu i} = \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i}. \tag{12}$$

However, the other Lagrange relation

$$\frac{\partial \dot{x}_\nu}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial x_\nu}{\partial q_i}\right)$$

does not exist due to the presence of non-holonomic relationships.

Upon noticing that

$$\frac{db_{\nu i}}{dt} = \sum_{j=1}^{n} \frac{\partial b_{\nu i}}{\partial q_j}\dot{q}_j + \sum_{s=n+1}^{u} \frac{\partial b_{\nu i}}{\partial q_s}\dot{q}_s + \frac{\partial b_{\nu i}}{\partial t}; \tag{13}$$

$$\frac{\partial \dot{x}_\nu}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial b_{\nu j}}{\partial q_i} \dot{q}_j + \frac{\partial b_\nu}{\partial q_i} \tag{14}$$

and upon subtracting (14) from (13) we will obtain an analogy of the second Lagrange relationship

$$\frac{d b_{\nu i}}{dt} = \frac{\partial \dot{x}_\nu}{\partial q_i} + \xi_{\nu i}, \tag{15}$$

where $\xi_{\nu i}$—is a certain expression whose form is self-evident.

We will write an expression for the kinetic energy of the system which expression has been transformed taking into consideration the above relationships:

$$T = \frac{1}{2} \sum m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2).$$

Using expressions (10), (11), (12), and (15) in the regular manner[4] we will obtain equations of motion in the form:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = R_i + Q_i, \; (i = 1, 2, \ldots n), \tag{16}$$

where $Q_i = \sum_\nu F_{\nu x} b_{\nu i}$ is the generalized active force, $R_i = \sum_\nu R_{\nu x} b_{\nu i} + \sum m_\nu x_\nu \xi_{\nu i}$ is the generalized force of reaction of the relationships.

The equations (16) can be called "the inherent equations" of motion of a non-holonomic system, since their left-side parts contain regular Lagrangian expressions, which are expressed by the projection of the acceleration of the describing point into the coordinates lines of a curvilinear system of coordinates of that multidimensional space in which the point, which represents the motion of the given system moves. Equations of this type were first published by MacMillan.[5]

The above solution is evidently not applicable to non-holonomic systems which are non-linear in respect to velocities, since it is impossible to obtain either the relationships (11), or the relationships (12) or (15) for this case.

Serious difficulties are encountered in the application of the D'Alembert-Lagrange principle to the solution of equations of motion of mechanical systems with non-linear non-holonomic relationships of the first degree and with relationships of the second degree.

If we directly apply the Holder principle to the relationships which are expressed by equations

$$f_j(\bar{r}_v, \dot{\bar{r}}_v, t) = 0, \tag{17}$$

where f_j are some arbitrary functions, that is having written the equation of the relationships in differential form, we will set it equal to zero and we will substitute $\delta\bar{r}_v$ for $d\bar{r}_v$, then we will obtain a relation non-linear in respect to $\delta\bar{r}_v$ and the application of the D'Alembert-Lagrange principle becomes impossible. In order to obtain relationships which will be linearized in respect to $\delta\bar{r}_v$, N. G. Chetayev has proposed to use the following relationships in conjunction with (12)

$$\sum_{v=1}^{3N} \frac{\partial f_v}{\partial \dot{x}_v} \delta x_v = 0. \tag{18}$$

Relationships (18) are already widely used by a number of authors, and they have made possible to obtain the generalization of many hypotheses of classical analytical dynamics for the case of systems with non-linear, non-holonomic relationships of the first order. Attempts are being made; however, to prove that there exist systems for which the relationships (18) do not apply (V. S. Novoselov).

It is possible to present a simple example for which the application of relationships (18) will result in a physcial contradiction.

We will assume that a relationship is imposed upon a moving particle in such a manner that the coefficient of its velocity must be constant (an automobile or a flying machine with its propulsive force regulated accordingly).

The relationships will be expressed by the equation

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=C$$

and equations (18) will have the form

$$\dot{x}\,\delta x+\dot{y}\,\delta y+\dot{z}\,\delta z=0$$

meaning that $\bar{v}\delta\bar{r}=0$, that is as permissible transformation can be considered only those which are at right angles to the velocity, comething which is not apparent at a given strictly kinematic relationships.

Vylkovish (Rumania) has suggested the following linearization of the Hölder equations for a system described by (12): "If the equations of the relationships can be represented in the form:

$$\sum_{\nu}\bar{\alpha}_{j\nu}\,d\bar{r}_\nu+\beta_j\,dt=0 \qquad (j=1, 2, \ldots l), \tag{19}$$

where the vectores α_js and the scalars β_j are functions of $\bar{r}_\nu$, $\dot{\bar{r}}_\nu$ and even of $\ddot{\bar{r}}_\nu$, then the following relationships are proposed for the permissible transformation

$$\sum_{\nu=1}^{N}\bar{\alpha}_{j\nu}\,\delta\bar{r}_\nu=0. \tag{20}$$

In this manner for the relationships

$$e^{\dot{x}}\dot{x}^2+e^{\dot{y}}\dot{y}^2+e^{\dot{z}}\dot{z}^2=C$$

for instance, the space of virtual transformations will, according to Vylkovich, be determined as follows:

$$e^{\dot{x}}\dot{x}\delta x+e^{\dot{y}}\dot{y}\delta y+e^{\dot{z}}\dot{z}\delta z=0.$$

and according to Chatayev.

$$(e^{\dot{x}}\dot{x}^2+2e^{\dot{x}}\dot{x})\delta x+(e^{\dot{y}}\dot{y}^2+e^{\dot{y}}2\dot{y})\delta y+(e^{\dot{z}}\dot{z}^2+e^{\dot{z}}2\dot{z})\delta z=0.$$

But then the equations of motion, obtained by using the D'Alembert-Lagrange principle will not be identical, and this is physically contradictory.[6]

There exist certain propositions of similar kind which relate to relationships, which are expressed by an equation of higher order.

In general, we have in the given case a new field of analytical dynamics which is in necessity careful investigation of all its basic postulates.

FOOTNOTES

1) Page 34. This article, with some changes, is the text of a paper which was read at the First All-Union Conference on theoretical and applied mechanics in January 1960.

2) Page 34. From here on only two-sided relationships are investigated, that is those relationships which are expressed by equations.

3) Page 37. The given principle is, obviously automatically applicable to the holonomic rheonomic relationships $f_j(x_\nu, t) = 0$, from where

$$\sum_\nu \frac{\partial f_j}{\partial x_\nu} dx_\nu + \frac{\partial f_j}{\partial t} = 0 \quad и \quad \sum_\nu \frac{\partial f_j}{\partial x_\nu} \delta x_\nu = 0.$$

4) Page 42. As in the case of solution of Lagrange equations of the second kind.

5) Page 42. V. D. Macmillan Dinamika Tverdogo Tela (Dynamics of Rigid Body) IL (Foreign Literature Publication House) 1951. The solution presented here is more detailed, and as to the author claims, more simple.

6) Page 45. The Hölder relationships simply do not exist for the given case.

CONTROLLED SYSTEMS AS SYSTEMS WITH NON-HOLONOMIC RELATIONSHIPS

By

V. V. Dobronravov

A controlled material system, a regulated process, can be considered as a combination of events (mechanical, electrical, magnetic and of other forms) taking place according to a predetermined program, according to which the determination of the events must take place at a specific time period and should have a specific character. So, for instance, a controlled flying machine should follow a precalculated, previously prescribed trajectory, according to a definite law and should perform certain maneuvers; in particular it should be stabilized in respect to orientation in space, this is especially important when speaking about a cosmic flying machine.

The achievement of a given program can be assured by internal or external influences, originating as a result of information which is expressed by means of these or other signals. In other words, the program for the functioning of a controlled system can be considered as a complex of a certain kind of relationship which are imposed upon the parameters of the system, and which determine the system's condition at any given instant. If, for instance, the given controlled system is represented by a mechanical system, then for its investigation, computation and planning it is entirely possible to apply as well developed approach of analytical dynamics, which will include the setting up and integration of equations of motion, computation of the reactions between relationships, in this case mechanical influences, which bring about the realization of the given relationships which are in accordance with the program; and which will also include the investigation of the character of motion of the

given controlled system, and in particular the examination of its stability etc.

If the given controlled system consists not only of mechanical elements but contains also constituent parts of other kind-electrical, magnetic, thermal, chemical etc.: that is, if the process which is carried out by the given system is expressed not only by mechanical motions and influences, but also by phenomena which are related to various other properties of matter, then it is necessary to include appropriate corrections and additions into the outlined plan of investigation, by introducing other physical parameters (physical coordinates of the plan). It is necessary to employ the pertinent assumptions of electrodynamics, physics, chemistry etc. in the setting up of the equations of the controlled system.

However, the general approach to such controlled systems can remain unchangeable; a controlled system can be considered as a system with given relationships which are expressed by means of previously formulated equations relative to the parameters of the system and their derivatives, and to consider the realization of the problem to come about as a result of special forces of reactions, the latter being not only mechanical one, with the forces being subject to the computations of reactions between the relationships in the widest sense of the word.

We will first examine a controlled system represented by particles, this system may also be a real one, such as a system of two controlled flying machines, which execute maneuvers relative to one another (for

instance in the problem of interception). Some other actual controlled systems can be reduced, with a known degree of systematic, to particle systems.

We will examine mechanical systems only. Let us have a system composed of N particles B_ν with masses m_ν (ν =1,2,...N). We will assume that non-holonomic relationships are imposed upon the system, and that they are expressed by means of finite relationships between the Cartesian coordinates of the particles of the system x_ν(ν=1,2,...3N)[1] having the form

$$f_h(x_\nu, t)=0 \qquad (h=1, 2, \ldots k) \tag{1}$$

and in addition, linear non-holonomic relationships which are expressed by means of non-integrable differential equations in respect to the coordinates of the system

$$\sum_{\nu=1}^{3N} A_{s\nu}\dot{x}_\nu + D_s = 0 \qquad (s=1, 2, \ldots p), \tag{2}$$

where $A_{s\nu}$ and D_s are analytical functions of x and t (but are not the functions of there derivates), are imposed.

All the relationships are rheonomic, that is they are time dependent, which is nearer to the actualities of automation.

Holonomic relationships (1) can also be written in differential form, by differentiating each eqaulity (1) in respect to time t. As a result the set of all the relationships will be given by a uniform group of relationships, which we will write in vector form for compactness:

$$\sum_\nu \bar{b}_{j\nu}\dot{\bar{r}}_\nu + b_j = 0 \qquad (j=1, 2, \ldots l=k+p). \tag{3}$$

The dynamic differential equations of motion can in this case be written directly, by dividing all the forces into the given active ones $\bar{F}_\nu$, and the unknown forces of reactions between relationships $\bar{R}_\nu$ (both of these forces may be internal or external):

$$m_\nu \bar{a}_\nu = \bar{F}_\nu + \bar{R}_\nu. \tag{4}$$

Having shown the equations of motion (4) and the equations of relationships (3) in coordinates form, we will have a system of 3N+1 differential equations with 6N unknown functions of time x_ν, $R_{\nu x}$.

In this manner, the problem of guidance can, in principle, always be solved, but it is undetermined, since the number of unknown functions is larger than the number of equations ($1 < 3N$). The given condition may even have a practical application, since it will permit us to to choose the controlled actions (reactions of relationships) in such a manner as to satisfy these or those optimal requirements, depending on the character of the actual problem. Only in the case of 1+3N does the system have a unique motion (unique mode of operation) with fully determinable control effects.

We will examine one more mode of operation which is known in analytic dynamics by the name of a system "with ideal relations." In this case the following condition is imposed upon the reactions netween the relationships: the sum of elementary works of all reactions between relationships must be equal to zero in any permissible transformation of the system, and by the concept of permissible transformations for a system with rheonomic relations we mean, in accordance

with Hölder's principle, the transformations that satisfy the conditions.[2]

$$\sum_{\nu=1}^{N} \bar{b}_{j\nu} \delta \bar{r}_\nu = 0 \qquad (j = 1, 2, \ldots l) \tag{5}$$

In the space of actual transformations, as has been shown in the previous article, such relationships are not ideal.

For the case of ideal relationships, the reactive forces must satisfy the relationships.

$$\sum_{\nu=1}^{N} \bar{R}_\nu \delta \bar{r}_\nu = 0. \tag{6}$$

The given conditions show that the reactions between relationships should in actuality be in the space of the vectors $\bar{b}_{j\nu}$, that is they should have the form

$$\bar{R}_\nu = \sum_{j=1}^{l} \lambda_j \bar{b}_{j\nu}. \tag{7}$$

In this case it is possible to determine, for the purpose of integration, all the reactions as functions of active forces, coordinates of the system, velocities, masses, time and the coefficient of coupling.

From the equations of motion of each of the system's particles we can find its acceleration, taking into consideration (7), as follows:

$$\bar{a}_\nu = \frac{1}{m_\nu} \bar{F}_\nu + \frac{1}{m_\nu} \sum_{j=1}^{l} \lambda_j \bar{b}_{j\nu}. \tag{8}$$

Differentiating the equations of couplings, we find:

$$\sum_{\nu=1}^{N} \bar{b}_{j\nu} \bar{a}_\nu + \sum_{\nu=1}^{N} \frac{d\bar{b}_{j\nu}}{dt} \dot{\bar{r}}_\nu + \frac{db_j}{dt} = 0. \tag{9}$$

Exposing new time derivatives and taking accelerations from (8) we will obtain l linear equations in respect to the multipliers λ_h

$$\sum_{h=1}^{l} A_{jh}\lambda_h + B_j = 0 \qquad (j = 1, 2, \ldots l), \tag{10}$$

where

$$A_{jh} = \sum_{\nu=1}^{N} \frac{1}{m_\nu} \bar{b}_{j\nu}\bar{b}_{h\nu}; \tag{11}$$

$$B_j = \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} [(\bar{v}_\mu \operatorname{grad}_\mu)\bar{b}_{j\nu}]\bar{v}_\nu +$$

$$+ \sum_{\nu=1}^{N}\left(\operatorname{grad}_\nu b_j + \frac{\partial \bar{b}_{j\nu}}{\partial t}\right)\dot{\bar{r}}_\nu + \frac{\partial b_j}{\partial t} + \sum_{\nu=1}^{N} \frac{1}{m_\nu} \bar{b}_{j\nu}\bar{F}_\nu; \tag{12}$$

$$(\bar{v}_\mu \operatorname{grad}_\mu) \equiv \left(v_{\mu x}\frac{\partial}{\partial x_\mu} + v_{\mu y}\frac{\partial}{\partial y_\mu} + v_{\mu z}\frac{\partial}{\partial z_\mu}\right).$$

The determinant $\|A_{jh}\|$ of the system (8) is known to be different from zero, since its diagonal elements are of the form

$$A_{jj} = \sum_{\nu=1}^{N} \frac{1}{m_\nu} b_{j\nu}^2. \tag{13}$$

Having determined λ_h from (10), we obtain the weight $\bar{R}_\nu$ by means of (7), and the equations of motions of the system by integration.

It can be seen from (12) that the reactions between the couplings depend on the character of the variations of the space of virtual transformations, which is determined by the set of the coefficient $b_{\nu j}$, from its transformation in the space, and its deformation. All this is characterized by both the local time derivatives, and by the convective members that enter into the expressions (12).

The proposed direct method of investigation of a controlled system of particles, obviously, remains in full force even with the presence of non-linear non-holonomic couplings of the form

$$f_h(x_\nu, \dot{x}_\nu, t) = 0, \tag{14}$$

where f_h are certain functions, analytical in all their arguments.

In our discussion we exclude the case of ideal couplings in respect to the virtual space of the system, since we have shown in the previous article that the concept of permissible transformation for systems with non-linear non-holonomic couplings does not have an accepted unique definition.

For the case of a system not made up of particles, that is of a system having a geometrical configuration, which taking into account a non-holonomic couplings is defined by the generalized coordinates $q_1, q_2, \ldots q_u$, it is necessary to employ equations of motions of systems with, generally speaking, non-holonomic couplings. But all the known types of equations of motion of systems with non-holonomic couplings pertain also to couplings which are also ideal (even the equations with Lagrangian undetermined multipliers).

Each one of these types of equations, like the equations of Appel, Chaplygin, Tsenov, Hamel and others, has played an important part in mechanics. But if we were to apply these types of equations to non-ideal couplings, that if is we were to introduce into them the reactions between couplings, they would turn out to be unduly complicated for the purpose of finding the desired functions.

The most simple equations of motion are those equations which were derived in the previous article, and which were called "the inherent equations of motion of non-holonomic systems."

We will assume that the following non-holonomic linear couplings are imposed upon a system having the coordinates q_1, q_2,....q_u

$$\sum_{j=1}^{n} b_{sj}\dot{q}_j + a_s = 0 \qquad (s=1,2,\ldots l). \tag{15}$$

Then from u generalized velocities $\dot{q}_i$ (i=1,2.....u) we will obtain only n=u-l independent velocities. The corresponding generalized coordinates we will also call independent. The inherent equations of motion of the system will have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = R_i + Q_i \qquad (i=1,2,\ldots n). \tag{16}$$

We are gettign a system of n+l differential equations with n+u unknown functions $q_1, q_2, \ldots q_u$, $R_1, R_2, \ldots R_n$.

The system is again undetermined, but it has a solution. The given undetermination can be useful in the solution of problems when this or that type of optimalization is required.

For the convenience of calculation the equations (16) can be rewritten in a form which explicitly contains all the generalized accelerations. For a scleronomic system these equations will have the form

$$\ddot{q}_s + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{ {ij \atop s} \right\}\dot{q}_i\dot{q}_j = \sum_{\tau=1}^{n}\tilde{a}_{\tau s}Q_\tau + \sum_{\tau=1}^{n}\tilde{a}_{\tau s}R_\tau, \tag{17}$$

where the braces express Kristoffel's symbols of the second kind, and

the coefficient $\check{a}_{rs}$ represents the elements of a matrix which is a reciprocal of the matrix of coefficient a_{rs} of a quadratic form which expresses the kinetic energy.

$$2T = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{rs} \dot{q}_r \dot{q}_s.$$

Differentiating the equations of couplings (15) we will obtain a system (again undetermined) of l+n equations for the finding of u+n unknown functions $q_1, q_2, \ldots q_u$, $R_1, R_2, \ldots R_n$.

For ideal couplings we will have the following in respect to a virtual space

$$R_r = \sum_{s=1}^{l} \lambda_s b_{sr},$$

and further computations will be made by analogy with (8)-(13).

If the controlled system will not be purely mechanical, then it is possible to introduce into the equations the respective relationships of electrodynamics, magnetodynamics, chemical, kinetics, thermodynamics etc.

The scheme for investigation of controlled systems with linear non-holonomic couplings, which was generally described, is the most systematic and logical and it guarantees correct results and solutions in the investigation of actual controlled system.

For the setting up of equations of motion of systems with non-linear non-holonomic couplings it is necessary, at present to employ only the general theorems of the dynamics of mechanical systems.

FOOTNOTES

1) Page 49. We will denote the Cartesian coordinates of all the particles by one letter x_ν.

2) Page 51. See previous article.

ON THE RELATIONSHIPS BETWEEN COMPOUND QUANTITIES AND THEIR ELEMENTS

By

V. V. Dobronravov

By having a number of quantities and by decomposing them into their component terms, it is possible in a number of cases to obtain relationships, generally speaking non-linear ones, between the given quantities and their elements. This takes place for systems of dimensionless quantities, which are in particular related to the sets of vectors under consideration, although the vectors may be of differing dimensions. Such relationships can be obtained by applying the methods of undetermined multipliers and by performing simple algebraic operations. Such relationships can serve as master formulas in the investigation of some given process, in particular in mechanics in the investigation of the motion of a point of a system of particles. Sometimes they make it easier to discover new facts in the field of these or other phenomena.

In a few previous publications dealing with the given problem the author has already introduced such relationships, for instance, the case of motion of a rigid body with one stationary point, and also in other field (for instance, in nuclear physics and general mechanics).[1] One of the relationships which were introduced in the above mentioned articles, that of the relationships between the kinetic energy of a rigid body, the kinetic moment of a rigid body and its angular velocity has helped to determine a new interesting fact in the geometry of the motion of a symmetrical gyroscope of Lagrange-Euler.

In the present article we present a number of relationships of similar structure between geometrical and kinematic elements of a particle as it moves in an elliptic orbit. We will for the time being, limit ourselves to the investigation of quantities in three-dimensional spaces.

Let x and y be the coordinates of the moving particle,

a and b the semiaxes of the ellipse

$\dot{x}$ and $\dot{y}$ the components of velocity

$\ddot{x}$ and $\ddot{y}$ the components of acceleration

r the distance from the orgin of coordinates to the moving particle

$\bar{r}$ the radius-vector of the particle

$\bar{v}$ the velocity vector

$\bar{w}$ the acceleration vector.

We will write the first relationship in the form

$$\begin{aligned}&\left[(\dot{x}\ddot{y}-\dot{y}\ddot{x})-\left(\frac{x\ddot{y}}{a^2}-\frac{y\ddot{x}}{b^2}\right)+\left(\frac{x\dot{y}}{a^2}-\frac{y\dot{x}}{b^2}\right)\right](r^2-1)+\\&+\left[\left(\frac{x\ddot{y}}{a^2}-\frac{y\ddot{x}}{b^2}\right)-(x\ddot{y}-y\ddot{x})+xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right]\left[\frac{1}{2}\frac{d(r^2)}{dt}-1\right]+\\&+\left[\left(\frac{\dot{x}y}{b^2}-\frac{\dot{y}x}{a^2}\right)-xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+(x\dot{y}-y\dot{x})\right][wr\cos(\bar{w},\bar{r})-1]=0.\end{aligned}\qquad (K_1)$$

In order to check the formulas we will substitute the coordinates expressions of the quantities for the quantities proper, noticing that $wr\cos(\bar{w},\bar{r})$ —is a scalar product of $\bar{w}\bar{r}=\ddot{x}x+\ddot{y}y.$

By performing all the algebraic operations and by taking into consideration the equations of the trajectory, we will prove the correctness of the relationship (K_1).

We will present this proof in detail. We will rewrite the expression (K_1) in an open manner, keeping in mind that $r^2 = x^2 + y^2$;

$$\frac{1}{2}\frac{d(r^2)}{dt} = x\dot{x} + y\dot{y};$$

$$(x^2+y^2-1)\left(\dot{x}\ddot{y} - \dot{y}\ddot{x} - \frac{x\ddot{y}}{a^2} + \frac{y\ddot{x}}{b^2} + \frac{x\dot{y}}{a^2} - \frac{y\dot{x}}{b^2}\right) +$$

$$+ (x\dot{x}+y\dot{y}-1)\left(\frac{x\ddot{y}}{a^2} - \frac{y\ddot{x}}{b^2} - x\ddot{y} + y\ddot{x} + \frac{xy}{b^2} - \frac{xy}{a^2}\right) +$$

$$+ (x\ddot{x}+y\ddot{y}-1)\left(\frac{\dot{x}y}{b^2} - \frac{\dot{y}x}{a^2} - \frac{xy}{b^2} + \frac{xy}{a^2} + x\dot{y} - y\dot{x}\right) = 0.$$

We will perform the multiplications

(See equation on 59 a)

The expression that has been obtained after simplification we will write in the form

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(\dot{x}\ddot{y} - \dot{y}\ddot{x} - x\ddot{y} + y\ddot{x} + x\dot{y} - y\dot{x}) = 0.$$

Since the particle moves in an ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, it follows that the relationship (K_1) has been removed.

We will write a few similar relationships (which we will call K-relationships), pertaining to the motion of a particle along an elliptical trajectory.

The second relationship has the form

$$x^2\dot{x}\ddot{y}-\ddot{x}x^2\dot{y}-\frac{x^3\ddot{y}}{a^2}+\frac{x^2y\ddot{x}}{b^2}+\frac{x^3\dot{y}}{a^2}-$$

$$-\frac{x^2y\dot{x}}{b^2}+y^2\dot{x}\ddot{y}-y^2\dot{y}\ddot{x}-\frac{y^2x\ddot{y}}{a^2}+\frac{y^3\ddot{x}}{b^2}+$$

$$+\frac{y^2x\dot{y}}{a^2}-\frac{y^3\dot{x}}{b^2}-\dot{x}\ddot{y}+\dot{y}\ddot{x}+\frac{x\ddot{y}}{a^2}-\frac{y\ddot{x}}{b^2}-\frac{x\dot{y}}{a^2}+\frac{y\dot{x}}{b^2}+\frac{x^2\dot{x}\ddot{y}}{a^2}-\frac{x\dot{x}y\ddot{x}}{b^2}-$$

$$-x^2\dot{x}\ddot{y}+\dot{x}x\dot{y}\ddot{x}+\frac{x^2\dot{x}y}{b^2}-\frac{x^2\dot{x}y}{a^2}+\frac{yx\dot{y}\ddot{y}}{a^2}-\frac{y^2\dot{y}\ddot{x}}{b^2}-y\dot{y}x\ddot{y}+y^2\dot{y}\ddot{x}+$$

$$+\frac{y^2x\dot{y}}{b^2}-\frac{xy^2\dot{y}}{a^2}-\frac{x\ddot{y}}{a^2}+\frac{y\ddot{x}}{b^2}+x\ddot{y}-\dot{y}\ddot{x}-\frac{xy}{b^2}+\frac{xy}{a^2}+\frac{x\ddot{x}\dot{x}y}{b^2}-\frac{x^2\dot{y}\ddot{x}}{a^2}$$

$$-\frac{x^2y\ddot{x}}{b^2}+\frac{\dot{x}^2\ddot{x}y}{a^2}+x^2\ddot{x}\dot{y}-xy\ddot{x}\dot{x}+\frac{y^2\dot{x}\ddot{y}}{b^2}-\frac{yx\dot{y}\ddot{y}}{a^2}-\frac{y^2x\ddot{y}}{b^2}+$$

$$+\frac{y^2x\ddot{y}}{a^2}+xy\dot{y}\ddot{y}-y^2\dot{x}\ddot{y}-\frac{\dot{x}y}{b^2}+\frac{\dot{y}x}{a^2}+\frac{xy}{b^2}-\frac{xy}{a^2}-x\dot{y}+y\dot{x}=0.$$

$$\left[\frac{1}{2}\frac{d(r^2)}{dt}-v^2\right]\left[ab(\dot{x}\ddot{y}-\dot{y}\ddot{x})-\left(\frac{b}{a}x\ddot{y}-\frac{a}{b}y\ddot{x}\right)+\right.$$
$$\left.+\left(\frac{b}{a}x\dot{y}-\frac{a}{b}y\dot{x}\right)\right]+\left[\frac{1}{2}\frac{d(v^2)}{dt}-v^2\right]\left[\left(\frac{a}{b}\dot{x}y-\frac{b}{a}\dot{y}x\right)-\right.$$
$$\left.-\left(\frac{a}{b}xy-\frac{b}{a}xy\right)+ab(x\dot{y}-y\dot{x})\right]+xy\left(\frac{a}{b}-\frac{b}{a}\right)v^2\Delta=0 \qquad (K_2)$$

and is verified by taking into consideration the equalities $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0;\quad \frac{x\dot{x}}{a^2}+\frac{y\dot{y}}{b^2}=0$

$$\Delta=ab\left[(\dot{x}\ddot{y}-\dot{y}\ddot{x})-(x\ddot{y}-y\ddot{x})+(x\dot{y}-y\dot{x})\right]$$

The following relationship has the form

$$ab\left(\frac{x\ddot{x}}{a^2}+\frac{y\ddot{y}}{b^2}-w^2\right)\left[(\dot{x}\ddot{y}-\dot{y}\ddot{x})-(x\ddot{y}-y\ddot{x})+(x\dot{y}-y\dot{x})\right]+$$
$$+(w^2-x\ddot{x}-y\ddot{y})\left[ab(\dot{x}\ddot{y}-\dot{y}\ddot{x})-\left(\frac{b}{a}x\ddot{y}-\frac{a}{b}y\ddot{x}\right)+\right.$$
$$\left.+\left(\frac{b}{a}x\dot{y}-\frac{a}{b}y\dot{x}\right)\right]+(w^2-\dot{x}\ddot{x}-\dot{y}\ddot{y})\left[\left(\frac{b}{a}x\ddot{y}-\frac{a}{b}y\ddot{x}\right)-\right. \qquad (K_3)$$
$$\left.-ab(x\ddot{y}-y\ddot{x})+\left(\frac{a}{b}-\frac{b}{a}\right)xy\right]=0.$$

The relationships that have been presented may be useful as control formulas in the tracking of a flight of a cosmic flying machine: by means of substituting into them data about the parameters of motion which are received by radio-telemetric methods, it will be possible to see clearly the numerical value of the total tracking error.

As a relationship from the field of hydro-mechanics we will consider the case of a liquid that flows around a sphere, assuming that the vectors of the vortices are directed along the radii of the sphere. As a result the following condition will be fulfilled

$$\bar{v}\,\bar{\Omega}=0,\quad (\bar{\Omega}-\text{вихрь})$$

vortex

$$v_x\Omega_x+v_y\Omega_y+v_z\Omega_z=0.$$

In addition another obvious condition has to be assumed that is

$$\bar{r}\bar{v}=0, \text{ т. е. } xv_x+yv_y+zv_z=0,$$

since $|\bar{r}|^2=x^2+y^2+z^2=\text{const}.$

For these conditions the following relationships will be satisfied:

$$(v_x^2+v_y^2+v_z^2)[\dot{v}_x(\Omega_y z-\Omega_z y)+\dot{v}_y(\Omega_z x-\Omega_x z)+$$
$$+\dot{v}_z(\Omega_x y-\Omega_y x)]+(\dot{v}_x v_x+\dot{v}_y v_y+\dot{v}_z v_z)[x(\Omega_y v_z-\Omega_z v_y)+$$
$$+y(\Omega_z v_x-\Omega_x v_z)+z(\Omega_x v_y-\Omega_y v_x)]=0, \qquad (K_4)$$

where v_x, v_y, and v_z are the components of the total velocity. Proof can be performed directly.

We will examine an example of more simple K-relationships. Let x, y, z be the coordinates of a moving particle.

v_x, v_y, v_z be the components of velocity

a_x, a_y, a_z be the components of acceleration.

Then the following relationships will be satisfied, which can be easily verified by the substitution $v^2=v_x^2+v_y^2+v_z^2$; $a^2=a_x^2+a_y^2+a_z^2$.

$$v^2[a_z(v_y y+v_x x)-a_y v_y z-a_x v_x z]=(xv_x+yv_y+zv_z)[a_z(v_x^2+v_y^2)-$$
$$-a_y v_y v_z-a_x v_z v_x]+(a_x v_x+a_y v_y+a_z v_z)[v_z(v_x x+v_y y)- \qquad (K_5)$$
$$-(v_x^2+v_y^2)z];$$

$$a^2[-yzv_y+(y^2+x^2)v_z-xzv_x]=(xa_x+ya_y+za_z)[yv_z a_y+$$
$$+xv_z a_x-a_z(yv_y+xv_x)]+(v_x a_x+v_y a_y+v_z a_z)[a_z(x^2+y^2)- \qquad (K_6)$$
$$-yza_y-xza_x]+[x(v_y a_z-v_z a_y)+y(v_z a_x-v_x a_z)+$$
$$+z(v_x a_y-v_y a_x)](ya_x-xa_y).$$

Now an example from the field of the theory of elasticity. Let

$e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{zx}$ — be the components of deformations

$X_x, Y_y, Z_z, X_y, Y_z, Z_x$ — be the components of stresses

E — be Young's Modulus

σ — be Poisson's coefficient.

Then the following relationship will be correct

$$\begin{gathered}
Ee_{xx}\Delta = [e_{xx} - \sigma(e_{xy} + e_{xz})]\{X_x(e_{yy}e_{zz} - e^2_{yz}) + Y_y(e_{yz}e_{xz} - e_{xy}e_{zz}) + \\
+ Z_z(e_{xy}e_{yz} - e_{yy}e_{zx})] + [e_{xy} - \sigma(e_{yy} + e_{yz})][X_x(e_{yz}e_{xz} - e_{xy}e_{zz}) + \\
+ Y_y(e_{xx}e_{zz} - e^2_{xz}) + Z_z(e_{xz}e_{xy} - e_{xx}e_{yz})] + \\
+ [e_{xz} - \sigma(e_{yz} + e_{zz})][X_x(e_{xy}e_{yz} - e_{yy}e_{yz}) + Y_y(e_{xy}e_{yz} - e_{xx}e_{yz}) + \\
+ Z_z(e_{xx}e_{yy} - e^2_{xy})],
\end{gathered} \qquad (K_1)$$

where

$$\Delta = e_{xx}(e_{yy}e_{zz} - e^2_{yz}) + e_{xy}(e_{xz}e_{yz} - e_{xy}e_{zz}) + e_{xz}(e_{xy}e_{yz} - e_{yz}e_{xz}).$$

In order to prove the above relationship it is sufficient to perform the following substitution on the left side[2]

$$e_{xx} = \frac{1}{E}[X_x - \sigma(Y_y + Z_z)].$$

A similar relationship can be obtained from (K_7) by a direct cyclical substitution; in the left parts Ee_{yy} and Ee_{zz} should be written, and corresponding substitutions be made in the right sides. Substituting in (K_7) and its analogues the components of deformation according to the well known formulas

$$e_{xx} = \frac{T}{E}; \quad e_{yz} = \frac{1+\sigma}{E}Y_z; \quad e_{zx} = \frac{1+\sigma}{E}Z_x; \quad e_{xy} = \frac{1+\sigma}{E}X_y$$

we obtain relationships between one set of components of stress and the basic coefficients of elasticity.

Now an example of a general type, which can be practically applied in a variety of fields.

We will assume that there exist two sequences of quantities: (a_1, a_2, $a_3$0 and t1, t_2, t_3) and it is possible to determine a direct reciprocal relationship between the two. For instance, the second number of the second sequence can be regarded as the "price" or the "weight" of the respective quantity of the first sequence. In the theory of information it is possible to assume

a_i as the number of symbols, which are transmitted through one channel.

t_i as the duration of each symbol, which make up some message.

We will denote the total duration of transmission by

$$T = a_1 t_1 + a_2 t_2 + a_3 t_3.$$

Then the following relationship will be satisfied

$$\begin{aligned} T[3t_1t_2t_3 - (t_1^3 + t_2^3 + t_3^3)] = (t_1t_2 + t_2t_3 + t_3t_1)\{a_1[t_1(t_2+t_3) - (t_2^2 + t_3^2)] + \\ + a_2[t_2(t_1+t_3) - (t_1^2 + t_3^2)] + a_3[t_3(t_2+t_1) - (t_1^2 + t_2^2]\} + \\ + (t_1^2 + t_2^2 + t_3^2)[a_1(t_3t_2 - t_1^2) + a_2(t_3t_1 - t_2^2) + a_3(t_2t_1 - t_3^2)], \end{aligned} \qquad (K_8)$$

which can also be proven directly. Relationships similar in structure to the relationship (K_8), can also be written for sequences consisting of n quantities each.

FOOTNOTES

1) Page 57. See Proceedings of MIKhM (Moscow Institute of Chemical Machinery) No 10 (2), 1950.

2) Page 62. N. I. Muskhelishvili, "Nekotoryye Zadachi Teorii Uprugosti" (Some Problems of the Theory of Elasticity), 1933, pages 60 and 62.

ON THE QUESTION OF THE BEHAVIOR OF CERTAIN GYROSCOPES

By

S. S. Tikhmenev

The phenomenon which is discussed in the following article was observed during the testing of a gyroscope (fig. 1) which consisted of a base (1), in the form of a sphere truncated along the plane surface DD_1; and of the cylinder (2) which was attached to this base. The center of gravity C was kept at a sufficient distance a from the center of the sphere O in the negative direction of the axis Ox_1 by means of a weight placed inside the sphere (see fig. 1).

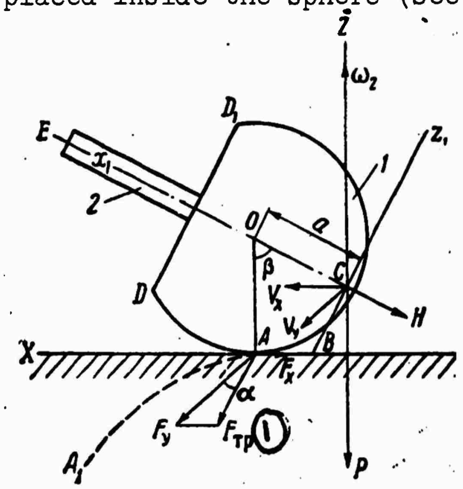

Fig. 1. Gyroscope of a special construction. 1) fr (subscript)

The following was discovered upon testing of this device. If the device is made to rotate at sufficiently high speeds around the axis Ox_1, and then is placed with its spherical surface on a horizontal plane, then, as the device turns around axis Ox_1 the end E begins to tilt more and more (that is the center of gravity C gradullly ascends) until the device stands on the end E of the cylinder (2), whereupon the center gravity C of the gyroscope continues to ascend, and as a result, the axis EC of the

gyroscope becomes almost vertical, whereupon the end E situates itself on the bottom and the center of gravity C on top.

The physical nature of this occurrence can be easily explained for the case when the center of gravity C of the instrument is situated at a sufficient distance from the center of the spherical surface and for a sufficiently large initial value of the angle β of inclination of the axis of the gyroscope to the vertical, namely at such an inclination that the point (in fig. 1 the plane of the drawing coincides with the plane of the gyroscope's inclination) of contact A of the spherical surface ond the supporting plane will be to the left (according to fig. 1) of the point B (the straight line BCZ which lies in the vertical plane of the drawing is a line perpendicular to the Ox_1 axis of the gyroscope).

In other words, we shall investigate the case where

$$\beta > \arccos \frac{a}{r},$$

where r is the radius of the spherical surface.

In this case the angle β should actually increase with time until the gyroscope turns over. We shall first prove this statement qualitatively.

Let us, for instance, examine the case where the kinetic moment of the gyroscope is directed towards the negative direction of axis Ox_1, it is evident that the reasoning that we shall quote below is equally applicable to the case where the direction of the vector of the kinetic moment will be

reverse, it will only be necessary to reverse the direction of the force F_y and of the velocity v_y in fig. 1.

When the gyroscope revolves with a high speed its spherical surface is unable to simply roll on the supporting plane and because of this it skids all the time. As a result, of this, for the given direction of the gyroscope's rotation, a horizontal frictional force F_{fr} directed almost perpendicular to the plane of the drawing and towards the observer will be applied to the point of contact A of the spherical surface (see fig. 1). This force creates a moment around axis Cz_1, whose vector is directed in the positive direction, and because of this the vector of the kinetic moment H turns in the direction of the indicated moment of internal forces relative to the axis Cz_1, that is the angle β increases.

For this case the equations of motion of the gyroscope can be approximately obtained in the following manner.

It is known that a similar gyroscope also moves in the horizontal plane for the conditions that have been given above, and its point of contact with the supporting plane moves along some curve AA_1 (see fig. 1). The projection of the center of gravity C of the gyroscope and the horizontal plane moves along a similar curve. In such a manner, the center of gravity of the gyroscope also has horizontal components of velocity v_x and v_y (see fig. 1), and besides this the gyroscope also has an angular velocity ω_z around the vertical axis Cx (at this point we will introduce, in addition to the coordinates system $CX_, y_{z1}$, which is associated with the axis of revolution of the gyroscope, another

system, Cxyz, in which the axis Cx is horizontal and the axis Cx vertical). Point A has a velocity $v_x+a\dot{\beta}\cos\beta$, in the direction of the x-axis and a velocity $v_y+a\omega_z\sin\dot{\beta}$. in the direction of the y axis. Since the gyroscope not only rotates about its axis, but also turns around axis y with the angular velocity $\dot{\beta}$, therefore, the sliding velocity of the point of contact of the gyroscope with the supporting plane in the direction of the x-axis turns out to be equal to $v_x+a\dot{\beta}\cos\beta-r\dot{\beta}$, and in the direction of the y-axis it is directed in the negative direction (as a result of the high speed of revolution of the gyroscope) is equal $-r\Omega\sin\beta+(v_y+a\omega_z\sin\beta)$, where Ω is the velocity of rotation of the gyroscope.

As a result of the above, the x and y components of the frictional force are equal to

$$F_x=-N\mu\sin\alpha,\quad F_y=N\mu\cos\alpha,$$

where N is the normal force at the point A and μ is the friction coefficient.

In this case

$$\operatorname{tg}\alpha=\frac{v_x+a\dot{\beta}\cos\beta-r\dot{\beta}}{r\Omega\sin\beta-a\omega_z\sin\beta-v_y}, \qquad (1)$$

and the normal force is given by

$$N=P\left(1+\frac{a\ddot{\beta}\sin\beta-a\dot{\beta}^2\cos\beta}{g}\right)\cong P.$$

Upon equating the forces along the x and y axes (in the Cxyz coordinates system) respectively, we obtain

$$
\left.\begin{aligned}
-P\mu\sin\alpha - P\frac{\dot{v}_x}{g} + P\frac{v_y\omega_z}{g} = 0 \\
P\mu\cos\alpha - P\frac{v_x\omega_z}{g} - P\frac{\dot{v}_y}{g} = 0,
\end{aligned}\right\} \qquad (2)
$$

and the abbreviated equations of motion of the gyroscope along the y and z axes (in the same coordinates system) will be

$$
\left.\begin{aligned}
-aP\sin\beta + P\mu\sin\alpha\,(r - a\cos\beta) + H\omega_z\sin\beta = 0 \\
P\mu\cos\alpha\,(a - r\sin\beta) - H\dot{\beta} = 0.
\end{aligned}\right\} \qquad (3)
$$

We have thus obtained five equations (1), (2) and (3) for the determination of the unknown v_x, v_y, ω_z, β and α, but a solution by means of solving a system of five simultaneous homogenous equations it quite difficult to obtain, and these equations are brought down only in order to provide additional clarification to the physical explanation of the substance of the phenomenon under investigation.

As the angle β increases constantly the device approaches the position where the end of cylinder (2) begins to touch the supporting plane. After this discussion, which was made in the form of an analogy, it is possible to present a discussion of the case where the device lowers itself onto the bearing plane not upon the spherical surface but upon some peripheral point on the end of the cylinder (2). In this case it should be kept in mind that also in this case the point of contact between the bearing surface and the cylinder (2) will keep on sliding.

On the basis of similar discussions it is possible to show

that the center of gravity C of the gyroscope will continue to ascend in this case also.

In this final stage of the gyroscope's motion the equations (2) remain unchanged, and the equations (1) and (3) take the following form

$$\left.\begin{aligned} &\operatorname{tg}\alpha=\frac{v_x+L\dot{\beta}\cos\beta-r_1\dot{\beta}\sin\beta}{r_1\Omega-(L\sin\beta+r_1\cos\beta)\,\omega_z-v_y}\\ &-P(L\sin\beta+r_1\cos\beta)+P\mu\sin\alpha\,(r_1\sin\beta-L\cos\beta)+\\ &\quad+H\omega_z\sin\beta=0,\\ &\quad P\mu L\cos\alpha-H\dot{\beta}=0,\end{aligned}\right\} \qquad (4)$$

where L=CE is the distance from the center of gravity of the gyroscope to the end of cylinder (2);

r_1 is the radius of curvatures of the end E of the cylinder (2).

In the equations (4) $\frac{\pi}{2}<\beta<\pi,$ and as a result $\cos\beta<0.$

In this manner the center of gravity C of the gyroscope can continue to ascend up to the point till the angle β will reach the value given by $\operatorname{arc\,tg}\left(-\frac{r_1}{L}\right),$ on the condition, of course, that the speed of rotation of the gyroscope should be

$$\Omega>\frac{v_y+(L\sin\beta+r_1\cos\beta)\,\omega_z}{r_1}.$$

After this, since the change in sign of the first member of the second equation (4) can only result in the change of sign of the velocities, ω_z and v_x (which in turn means that there will be consequent changes in sign of the second term of the second equation (4) and of the first term of the first equation (2), but in this case, according to the third equation (4), the sign of $\dot{\beta}$ does not change.

One may add that in the equations (1) and (3) as well as in the equations (4) the speed Ω of the gyroscope's revolution, and as a result, its kinetic moment H are not constant, since Ω constantly decreases from the beginning on, approximately according to the equation

$$\dot{\Omega} = -\frac{P\mu r \sin\beta + k\Omega^2}{J_0},$$

where k is some dimensional coefficient, which depends on the dimensions and shape of the gyroscope and on the density of the surrounding air;

J_0 is the moment of inertia of the gyroscope relative to its axis of revolution, starting from the time when the gyroscope leans on the end of the cylinder (2)

$$\dot{\Omega} = -\frac{P\mu r_1 + k\Omega^2}{J_0}.$$

In this manner, during both stages of the gyroscope's rotation we will have a system consisting not of five but of six equations.

Behavior similar to the one described above will, apparently, be inherent to many systems of similar construction, if these systems will be given a high angular velocity around their axes of symmetry.

THREE THEOREMS ABOUT THE STRONG MINIMUM IN THE CLASSICAL PROBLEM OF THE CALCULUS OF VARIATIONS

By

V. F. Krotov

We are examining the problem of a strong minimum of the function

$$I=\int_a^b F(x, y_1, \ldots, y_k, y_1', \ldots, y_k')\,dx \qquad (1)$$

for the condition

$$\left.\begin{aligned} y_1(a)&=a_1, \ldots, y_k(a)=a_k;\\ y_1(b)&=b_1, \ldots, y_k(b)=b_k,\end{aligned}\right\} \qquad (2)$$

where $y_i(x)$ $(i=1,2\ldots\ldots,k)$ are continuous functions having a piecewise continuous derivative, that is $y_i(x)\in C_1$, and belong to a certain closed region of a B (k+1) dimensional space $(x,y_1,y_2\ldots\ldots y_k)$, in which space the function $F(x, y_1, \ldots, y_k, z_1, \ldots z_k)$ and its partial derivatives is continuous in respect to all the arguments for any value of $z_1\ldots\ldots,z_k$.

We will derive some results that apply to this problem by using the theory of discontinuous solutions of the problems of the calculus of variations which were developed in /1/, /2/, and /3/. We will study the reciprocal coupling of discontinuous extrema with the extrema of class C_1 which coupling has, besides its application to the problem under consideration, a significance of its own.

It turns out that the existence and the character of the minimum in class C_1 is related to the character and position of the discontinuous extrema studies in /1/ and /2/. By using this relation we derive the necessary conditions for the

existence of a strong minimum of the I function which couplings possess, in case they are applicable, substantial advantages compared to the Weierstrass condition, namely: first, they are significantly simpler than the latter and permit in many cases to come to the conclusion that a strong minimum is absent apriori, that is without finding the equation of the extremum, when the Weierstrass condition in this case when these requires that the extremum be known; secondly, in the case when these relationships are applicable, these conditions are stronger than the Weierstrass condition, that is, in those cases when the latter are fulfilled, the former may not.

We examine examples which illustrate the application of these new necessary conditions. For the sake of exactness, all the discussions pertain to the minimum but it is obvious that they apply to the problem of a maximum also, if we only change F to -F in the respective formulas.

We will start with the examination of a functional having one unknown function:

$$\left.\begin{aligned} I&=\int_a^b F(x,y,y')\,dx; \\ y(a)&=a_1; \quad y(b)=b_1. \end{aligned}\right\} \qquad (3)$$

We will extend this functional over a wider number of U functions which may have any large but finite number of discontinuities of the first kind x_j $(j=1, 2, \ldots, \mu)$ over the region /a, b/, and which have a piecewise continuous and limited derivative y'(x) over each of the i intervals (a, x_1), $(x_1, x_2), \ldots, (x_{\mu-1}, x_\mu)$, (x_μ, b).

We will additionally define the functional (3) of this set by the formula:

$$I(u) = \lim_{m \to \infty} I(u^m), \tag{4}$$

Where $u^m \in C$, is a line which coincides always with u with the exception of sufficiently small regions of the points x_j, in which the vertical intervals are substituted for by intervals inclined to the vertical by the angle 1/m.

We will assume m<0 for clockwise inclination and m<0 for counterclockwise inclincation.

From the single valuedness of y_m (x) it follows that

$$\operatorname{sign} m = \operatorname{sign}[y(x_j+0) - y(x_j-0)]. \tag{5}$$

The properties of the extremum of functional (3) of the set U were studies in /1/ and /2/. We will give a short review of the results of these papers which will be necessary in the present investigation.

If in the entire B region of the xy plane there exist limits

$$W(x, y, \operatorname{sign} m) = \lim_{m \to \pm\infty} F(x, y, m) \frac{1}{m} \tag{6}$$

and in addition

$$W(x, y, 1) = W(x, y, -1) \tag{7}$$

(functionals of the SH type) then the minimal of the functional (3) in the set U (in general not belonging to the set) has the property

of local independence of its elements from the neighboring elements (and consequently from the ends) and also from the local minimum of the function

$$S(x, y, z)=F(x, y, z)-W(x, y)z-\int_{c}^{y} W_{x}(x, \xi)\, d\xi, \qquad (8)$$

where c is an arbitrary constant and y (x) and z (x) are independent functions.

The last property permits us to write the final equation of such an extremum in the form of a necessary condition of the extremum of a function of two variables S(x, y, z,) for each chosen value of $x \in [a, b]$.

$$\left.\begin{aligned} S_y &\equiv F_y - W_y z - W_x = 0; \\ S_z &\equiv F_z - W = 0. \end{aligned}\right\} \qquad (9)$$

Each solution of the system (9) y^0 (x) and $z^0(x)$ along which S(x,y,z) is minimal for each chosen value of x in the region

$$|y-y^0|<\varepsilon; \; |z-z^0|<\infty \qquad (10)$$

gives a relative minimal $\bar{u}$ in the U class,

This means that if we will take a set of broken lines $\{\gamma_n\} \in U$ which possess not more than n points of discontinuous and which are given by the equation

$a=x_1<x_2< \ldots <x_{n-1}<x_n=b$

$$\left.\begin{aligned} y(x) &= y_i + y_i'(x-x_i) \quad x_i<x<x_{i+1} \qquad (i=1, \ldots, n-1) \\ y_i &= y(x_i+0); \quad y_i'=C_i \end{aligned}\right\} \qquad (12)$$

(C_i are constants, in addition $|x_{i+1}-x_i|_{max} \to 0$ at $n \to \infty$ and taking

$$y_i = y^0(x_i); \quad y_i' = z^0(x_i),$$

then at $n > N$

$$I(u) > I(\gamma_n),$$

where $\bar{u}$ is any previously assigned line which belongs to the set U and to the region (10).

If

$$z^0(x) = y^0(x) \qquad (13)$$

holds everywhere with the exception of a finite number of points, then the minimal $\bar{u}$ in the U class itself belongs to this set. If in addition $y^0(x)$, $z^0(x)$, and

$$y^0(a) = a_1, \; y^0(b) = b_1, \qquad (14)$$

then $\bar{u} \in C_1$ and it coincides with the solution of the Euler equation of the limiting conditions (3). It is obvious that in this case the solution of the Euler equation is obtained directly from (9) in the form

$$y(x) = y^0(x) \text{ и } y'(x) = z^0(x).$$

By the term extermal we will understand a function which satisfies the first necessary condition of an extremum.

In the C_1 class this condition is met by the Euler equation and the Erdman conditions in the corner points, and in the U class the equations (9) etc.

If the limits (6) exist, but they are not equal to one another then the function W, regardless of x and y is dependent on the sign

of the difference y'-z. This results in the fact that the functional (3) can have, in addition to the above described minimals, also minimals in the class U which coincides with the usual Eulerian minimals in the class C_1, which do not possess the properties of local independence and local minimum, and consquently do not satisfy the system (9).

In addition along these minimals it is necessary to satisfy additional necessary conditions in the form of a minimum of the function

$$S[x, y, z, \operatorname{sign}(y'-z)]$$

that is at $-\infty < z < \infty$.

$$\begin{aligned} \Delta S = S[x, y, z, \operatorname{sign}(y'-z)] - S(x, y, y', \operatorname{sign}(y'-z)] = \\ = F(x, y, z) - F(x, y, y') - (z-y') W[x, y, \operatorname{sign}(y'-z)] \geqslant 0. \end{aligned} \tag{15}$$

If the limits (6) exist not over the entire /a,b/ interval, but only in a finite number of points $x=\mu_i (i=1, 2, \ldots, r)$ (functionals of the types II and III), then:

1) if $W(\mu_i, y, 1) = W(\mu_i, y, -1)$ then the extremal y (x) in the class U consists of continuous intervals, which are connected by means of vertical segments at the points μ_i. The positions of the ends of the continuous segments of the extremal on the vertical line $x=\mu_i$ is given by the conditions

$$F_{y'} - W|_{x=\mu_i-0} = 0; \quad F_{y'} - W|_{x=\mu_i+0} = 0. \tag{16}$$

In a particular case, $y(\mu_i-0) = y(\mu_i+0)$, then the extremal is continuous;

2) if $W(\mu_i, y, 1) \neq W(\mu_i, y, -1)$ then there exists in the U class a continuous extremum which coincides with the extremum of the C_1 class, if at the points μ_i of this extremum there exists the condition at $\eta > 0$

$$\{F_{y'}[\mu_i, \bar{y}(\mu_i), y'(\mu_i)] - W[\mu_i, y(\mu_i), \text{sign}\, \eta]\}\, \eta \geqslant 0.$$

Finally, if the limits (6) do not exist everywhere over the interval /a,b/, (functionals of the I type) then the extremums in the U class coincides with the extremums in the C_1 class.

Interrelationships of the Extrema in the Class of Continuous and Discontinuous functions

The existence and character of the minimum in the C_1 class depends on the position and character of the minima in the U class. This interrelationship is of definite interest by itself, and in addition it permits to establish new necessary condition for the minimum.

THEOREM 1. In order that the line $l \in C_1$ should give a strong minimum to the functional (3) in the C_1 class it is necessary and sufficient that it should be a minimum in the U class.

The sufficiency is obvious since $C_1 \in U$. We will prove the necessity. We will assume the opposite, that is, that there exists such a line $u \in U$ in the given region of the zero order of line l that

$$I(l) - I(u) = \Delta > 0. \qquad (17)$$

From definition (4) we have

$$I(u) = I(u^m) + \varepsilon(m), \tag{18}$$

where $u^m \in C_1$ for any fixed value of m and $|\varepsilon| \xrightarrow[m\to\infty]{} 0$.

Since $u^m \in C_1$ then

$$I(u^m) > I(l) > I(u)$$

and, consequently, $\varepsilon(m) < 0$. We will choose such an N that for m=N

$$|\varepsilon(m)| < \frac{\Delta}{2}. \tag{19}$$

From (17), (18) and (19) we have:

$$I(l) - I(u^m) = \frac{\Delta}{2} > 0$$

$$u^m \in C_1,$$

which contradicts the hypothesis. The theorem is proven.

This theorem establishes the relationship between the minima in the U class and the C_1 class, and also gives a number of convenient necessary conditions for the existence of the minimum of the functional (3) in the C_1 class. We will formulate these results in the form of conclusions.

Conclusion 1. If the limits (6) do not exist everywhere in the B region (functionals of the I type), then the minima in the U and C_1 classes coincide and consequently the functional (3) in the C_1 class can possess both a weak and strong minimum.

Conclusion 21 If the limits (6) exist at least at one vertical line

$$x = x_0 \in [a, b], \; y \in B$$

and in addition, $W(x, y, 1) = W(x, y, -1)$, then the extrema in the classes U and C_1 do not coincide, in which case the functional (3) can have only a weak minimum on the extremum of the C_1 class.

Note 1. The conclusion 2 contains an unproven hypothesis, namely, that in the C_1 only a weak minimum can take place. In order to fill this gap, it is sufficient to prove the existence of a weak minimum in any particular problem. We will examine the functional (see /3/, page 114 and 160, problem 11):

$$I = \int_0^b \frac{dx}{y'}; \quad y(0) = 0; \quad y(b) = b_1; \qquad b > 0; \quad b_1 > 0. \tag{20}$$

For $|m| \to \infty$ we have

$$W(x, y, 1) = W(x, y, -1) = \lim \frac{1}{m^2} = 0 \tag{21}$$

over the entire xy plane that is, the functional (20) belongs to the type under consideration. The equation of the extremum in the class C_1 (solution of the Euler equation) is:

$$y = \frac{b_1}{b} x. \tag{22}$$

It is easy to check that on this extremum there are satisfied the additional conditions of a weak minimum. Consequently, the existence of a weak minimum in the C_1 class of functionals of the given type on the usual Eulerian extremums is actually possible.

Note 2. The formulation of the conclusion 2 contains an intentional unexactness, namely, the minima of the functionals of the type under consideration may coincide, however, this coincidence is an exception,

similar to the existence of extrema in the class C_1 in the special case Fyy=0. The basic fact consists in the fact that in the given case the extrema in the classes U and C_1 are given by entirely different necessary conditions; for instance, the solution of the Euler equation for the limiting conditions (3) in the first case and the local minimum S(x,y,z) for a fixed x in the second case; due to this fact the non-coincidence of the extrema is a rule, and their coincidence is an accidental aposteriori fact, when for instance, the given points (a,a_1) and (b,b_1) turn out to be lying on a line of zero nearness to the minimum in the U class /condition (14)/ and in addition the equality (13) (for functionals of the IV type) will a posteriori turn out to be correct. For functionals of the II type this is possible only in such special cases as the aposteriori coincidence $\bar{y}(\mu_i-0)=\bar{y}(\mu_i+0)$, or the non-uniqueness of the solution of the Euler equation.

Namely, in order to underline this fact we write that the extrema do not coincide, keeping in mind, however, the above remark.

Conclusion 3. If the limits (6) exist at least on one vertical line $x=x_0\in[a, b], y\in B$ and in addition $W(x, y, 1)=W(x, y, -1)$ than the extremum in the C_1 class coincides with one of the extrema in the U class and the functional (3) can have on it both a strong and a weak minimum in the C_1 class. In addition a strong minimum is reached only in the case when on the segments where (6) exists the conditions (15) and (16) are satisfied, and in the isolated points $x=x_0\in[a, b]$ — where (6) exists the condition (16') is satisfied.

The term extremal here denotes a line which satisfied the first necessary condition for the existence of an extremum (in contrast with the preceeding case, here this is represented in both classes by the Euler equation with corresponding fringe conditions).

Necessary Conditions for the Existence of a Strong Minimum

We will prove two more theorems which give the necessary conditions for the existence of a strong minimum of the functional (1) in the C_1 class.

We will denote

$$\left.\begin{aligned} & W_i(x, y_i, \text{ sign } m) = \lim_{|m| \to \pm\infty} \frac{1}{m} F[x, \bar{y}_1(x), \ldots, y_i(x), \bar{y}_k(x); \\ & \bar{y}'_1(x), \ldots, \bar{y}'_{i-1}(x), \frac{1}{m}, \bar{y}'_{i+1}(x), \ldots, \bar{y}'_k(x)]; \\ & S_i(x, y_i, z_i) = F[x, \bar{y}_1(x), \ldots, y_i, \ldots \\ & \ldots, \bar{y}_k(x), \bar{y}'_1(x), \ldots, z_i, \ldots, \bar{y}'_k(x)] - W_i z_i - \\ & - \int_c^{y_i} W_{ix}[x, \xi, \text{ sign } (y'_i - z_i)]\, d\xi, \end{aligned}\right| \qquad (23)$$

where c is an arbitrary constant;

$$W_{ix} = \frac{\partial W_i}{\partial x} + \sum_{j \neq i} \frac{\partial W_i}{\partial y_j} \bar{y}'_j(x) + \sum_{j \neq i} \frac{\partial W_i}{\partial y'_j} \bar{y}''(x). \qquad (24)$$

THEOREM 2. Let $\bar{y}(x) = [\bar{y}_1, \ldots, \bar{y}_k]$ — be an extremum of the functional (1) in the C_1 class of lines. If at least in one isolated value $x = x_0 \in [a, b]$ there exists even one such an i, that in as much as desired small region

$$|y_i - \bar{y}_i(x_0)| < \varepsilon; \qquad (25)$$

there exists a pair of functions

$$W_i(x_0, y_i, \pm 1),$$

and in addition either

$$1) \quad W_i(x_0, y_i+1) = W_i(x_0, y_i, -1)$$

in the region (25) and

$$F_{y_i'}[x_0, \bar{y}(x_0), \bar{y}'(x_0)] - W_i[x_0, \bar{y}(x_0)] \neq 0, \tag{26}$$

or

$$2) \quad W_i(x_0, y_i, 1) \neq W_i(x_0, y_i, -1)$$

and the inequality

$$\{F_{y_i'}[x_0, \bar{y}(x_0), \bar{y}'(x_0)] - W_i[x_0, \bar{y}(x_0), \bar{y}_i(x_0), \operatorname{sign}\eta]\}\eta \geq 0 \tag{27}$$

at $\eta \neq 0$ has no place, then the functional (1) on the line y(x) does not possess a strong minimum in the set C_1.

Let the conditions (26) be satisfied at one point $x_0 \in [a,b]$. At k=i=1 the functional (1) belongs to the II type and the conditions (26) signify the absence of continuous minima over the set U of continuous functions, which belong to the ε of the region of zero order of the function $\bar{y}$ (x). It follows from the theorem 1 that in this case the minimum in the C_1 class over $\bar{y}$ (x) does not exist. If k>1, then by fixing a certain i and by considering the functions $y_j = \bar{y}_j(x)$ $(j \neq i)$ to be given, we will not obtain the functional I_1/y_i (x)/, which is only dependent on one function $y_i(x)$, belonging to the II type and consequently not having a strong minimum over C_1.

But this means that in as much as desired small region the set $\bar{y}$ (x) there exists a set $y(x) = [\bar{y}_1(x), \bar{y}_2, \ldots, y_i(x), \ldots, \bar{y}_n(x)]$, which is not identically equal to the former, and is such that $I(y) < I(\bar{y})$, that is, functional (1)

does not have a strong minimum over y (x). The case (2) is proven in an analogous manner. The theorem is proven.

Remark 1. if along the line of extremum y (x) the condition $F_{y'y'} \neq 0$ is fulfilled, then the unfulfilment of the condition (26) is only possible for certain specially selected, single-valued values for this functional $a_1=\bar{a}_1$ and $b_1=\bar{b}_1$. Due to the fact, if the limits (6) exist at the point x_0 and are equal then the existence of a minimum is scarcely probable.

On the other hand, if the border points (a,a_1) and $b,b_1)$ are given in the general form, then in order that a strong minimum be absent it is sufficient that the limits (6) should exist and be equal to one another at least for one value of $x_0 \in [a,b]$.

THEOREM 3. Let $\bar{y}(x)=[\bar{y}_1, \ldots, \bar{y}_k] \in C_1$ -- be the line of extremum of the functional (1) in the C_1 class. If there exists as much as desired small but fringe segment $[\alpha, \beta] \in [a, b]$, on which there exists at least one pair of functions $W_i[\bar{x}, \bar{y}(x), y_i, \pm 1]$ in as much as desired small ε-region of zero order B_ε function $\bar{y}_i(x)$ over the segment $[\alpha,\beta]$ and in addition

$$W_i[x, \bar{y}(x), y_i, 1] = W_i[x, \bar{y}(x), y_i, -1],$$
$$x \in [\alpha, \beta], \; y \in B_\varepsilon,$$

then the functional (I)does not have a strong minimum over y(x) in the C_1 class, if one of the following conditions is satisfied

$$\text{a) } y^0(x) \in C_1; \; z^0(x) =/= y^{0\prime}(x); \quad x \in [\alpha, \beta], \tag{28}$$

where y^0 (x) and z^0 (x) are functions found from the conditions

$$S_i[x, \bar{y}(x), y^0, z^0] = \min S_i[x, \bar{y}(x), \xi, \eta]; \quad \xi \in B_\varepsilon; \; -\infty < \eta < \infty \tag{29}$$

for each fixed value of x;

$$б)\ y^0(\alpha) \neq \bar{y}(\alpha), \tag{30}$$

or

$$y^0(\beta) \neq \bar{y}(\beta).$$

Remark 1. It should be emphasised that $\bar{y}_1(x)$ does not enter into condition a). This very often permits to solve the problem of a strong minimum apriori, that is, without finding the equation of the extremum $y=\bar{y}(x)$.

Remark 2. Conditions a) and b) are almost all inclusive, the unfulfilment of which is an exception (see also remark 2 to theorem 1). Due to this fact the basic meaning in answering the question of the existence of a strong minimum is inherent in the first condition, that is, the existence and equality of the functions $W_i(x, \bar{y}(x), y_i, \pm 1)$ over the segment $[\alpha, \beta]$. If this condition is fulfilled, then the existence of a strong minimum is scarcely believable.

We will prove the theorem 3. Let k=i=1. We will present functional (I) in the form

$$I[y(x)] = I_{a\alpha} + I_{\alpha\beta} + I_{\beta b}. \tag{31}$$

here the subscripts denote limits of integration.

The functional $I_{\alpha\beta}$ belongs to the type IV in the region B_ε and consequently its line of minimum in the U class is independent of the end points $y(\alpha)$ and $y(\beta)$.

Let $\bar{y}(x)$ be a strong minimum of I(y). A segment of this line of minimum over $[\alpha, \beta]$ should be a strong minimum of $I_\alpha(y)_\beta$ for the fringe

conditions $y(\alpha)=\bar{y}(\alpha-0)$; $y(\beta)=\bar{y}(\beta+0)$.

In accordance with theorem 1, $\bar{y}(x)$ will be a strong minimum of over the segment $[\alpha_1, \beta]$ only in the case when it coincides with the minimum $I_{\alpha\beta}$ in the U class. But this coincides is not possible, since due to the condition a) the latter is not piecewise smooth and consequently does not belong to the C_1 class. On the other hand, if the condition a) is not fulfilled, then the coincidence is impossible by virtue of condition b), since in this case the minimal in the U class does not satisfy the fringe conditions $y(\alpha)=\bar{y}(\alpha-0)$ and и $y(\beta)=\bar{y}(\beta+0)$ Consequently the section $\bar{y}$ (x) does not give a strong minimum of $I_{\alpha\beta}$ over segment $[\alpha, \beta]$ which in turn means that $\bar{y}$ (x) also does not give a strong minimum I.

Theorem is proven for k=1. For k>1 it is only necessary to repeat the proof of theorem 2.

Conclusions

Theorems 2 and 3 give the necessary conditions for the existence of a strong minimum of functional (1). We will note the main advantages of these conditions in comparison with the necessary condition due to Weierstrass.

$$E(x, \bar{y}, \bar{y}', \eta) \geqslant 0 \qquad -\infty < \eta < \infty.$$

1. These conditions, in those cases where they are applicable, are stronger, that is, they point to the absence of a minimum in the time when the Weierstrass condition is fulfilled (see below, example 3).

2. These conditions are simple, especially in the case of arbitrary boundary conditions, when in order to conclude that a minimum is absent it is only required that the functions $W_i[x, \bar{y}(x), y_i, \pm 1)$, should exist and be equal to one another at least for one i and at least for one $x = x_0 \in [a, b]$ (see remark 1 to theorem 2 and remark 2 to theorem 3).

3. These conditions permit to give a conclusion about the presence of a strong minimum apriori, that is, without finding the equation of the line of extremum, while Weierstrass condition (as well as the Legendre condition) require previous finding of the extremum in these cases (see below, example 1 and 2).

We will examine a few example, which illustrate the application of the obtained conditions and their relation to the Weierstrass condition.

Example 1. To give a conclusion about the strong extremum of the functional

$$\left.\begin{array}{l} I = \int\limits_a^b \left\{F_1(x, y_1, \ldots y_k, y_1', \ldots, y_{k-1}')\, f(x) + e^{-y_k'}\right\} dx; \\ y_j(a) = a_j;\ y_j(b) = b \qquad (j = 1, 2, \ldots, k) \end{array}\right| \tag{32}$$

f (x) has a root $x = x_0 \in [a, b]$.

We have

$$W_k(x_0, y_k, 1) = W_k(x_0, y_k, -1) = \lim_{|m| \to \infty} \frac{1}{m} e^{-m} = 0;$$

over the entire xy_k plane

$$F_{y_k'} - W_k|_{x=x_0} = e^{-y_k'} \neq 0$$

not for any value of y'_k.

Consequently, the functional (32) satisfies the condition of the theorem 2. There is no strong minimum.

We will now set up the Weierstrass condition for the point x_o:

$$E|_{x=x_0} = e^{-\eta} - e^{-\bar{y}'_k(x_0)} + e^{-\bar{y}'_k(x_0)}[\eta - \bar{y}'_k(x_0)] \geqslant 0.$$

Nothing can be said about the correctness of this inequality without knowing the function $\bar{y}_k$ (x), that is, the equation of the line of extremum of the functional (32).

In this manner the theorem 2 permits to establish the absence of a strong minimum (and maximum) of the functional (32) apriori, while the Weierstrass condition requires prior knowledge of the extremum $\bar{y}$ (x). As far as the Legendre condition $F_{y'_k y'_k} \geqslant 0$, is concerned, it is fulfilled in the point x_0 $F_{y'_k y'_k}\big|_{x=x_0} = e^{-y'_k} \geqslant 0$ for any y'_k and consequently it is too weak.

Example 2. To come to a conclusion about the possibility of existence of a strong minimum of the functional

$$\left. \begin{aligned} I &= \int_a^b \frac{1+y^2}{y'^2}\,dx; \\ y(a) &= a_1; \qquad y(b) = b_1. \end{aligned} \right| \tag{33}$$

We have

$$W(x, y, 1) = W(x, y, -1) = \lim_{m \to \pm\infty} \frac{1+y^2}{m}\,\frac{1}{m} = 0$$

over the entire xy plane.

In accordance with the remark 2 to theorem 3 it is scarcely

believable that a strong minimum exists. We will check the condition 1 of theorem 3. We have $S(x,\xi,\eta)=\frac{1+\xi^2}{\eta^2}$. Since $S \geqslant 0$ for any value of ξ and η, then

$$S(x, y^0, z^0) = \inf_{-\infty<\xi,\ \eta<\infty} S = 0$$

This value takes place at $z^0(x)=\eta=\infty$ and for an arbitrary $y^0(x)=\xi$. Since the line of extremum is $\bar{y}(x) \in C_1$, then the condition $y^{0\prime}(x)=z^0(x)$ is unfulfillable for any value of $y^0(x)$ and consequently, in accordance with theorem 3 a strong minimum does not exist.

Functional (33) has an extremum

$$\bar{y}(x) = sh(C_1 x + C_2), \tag{34}$$

where C_1 and C_2 are found from the conditions

$$a_1 = \mathrm{sh}(C_1 a + C_2);\ b_1 = \mathrm{sh}(C_1 b + C_2),$$

(see /4/ pp 114 and 160, problem 8). Over which it reaches a weak minimum. By knowing the equation of the extremum (34) it is possible to prove the absence of a strong minimum by means of the Weierstrass condition. This condition, as in example 1, does not give an apriori result.

Example 3. To come to a conclusion about the existence of the minimum of the functional

$$I=\int_a^b xy^2 f(y')\,dx;\quad y(a)=0;\quad y(b)=0. \tag{35}$$

On the straight line $y(x)=0$ we have $=0$ for $a \leqslant x \leqslant b$.

Here $f(\xi)$ has the following properties:

1) f (ξ) is an even function; 2) f (ξ) $>$ M over $[-\infty\ +\infty]$ where $M<\infty$; 3) f (0) =0, f $(\xi) > 0$ for $\xi \neq 0$; 4) f' (ξ) exists and ξ is continuous over the entire real axis and f' (ξ) =0 everywhere except $\xi \equiv 0$. The line y(x) =0 is a line of extremum of the functional (35) and the absolute minimum is reached on it for $a > 0$.

For $a<0$ a minimum does not exist. We will check this fact with the help of two necessary conditions for the existence of a minimum the Weierstrass condition and the theorem 3.

The Weierstrass function $E \equiv 0$; consequently, the Weierstrass oondition $E \geqslant 0$ is fulfilled.

We have further

$$W(x, y, 1) = W(x, y, -1) = 0$$

over the entire xy plane. Further

$$S(x, \xi, \eta) = x\xi^2 f(\eta)$$

$$\inf S = \begin{cases} -\infty, & a \leqslant x < 0; \\ 0, & 0 \leqslant x \leqslant b; \end{cases}$$

$$-\infty < \xi, \eta < \infty$$

$$y^0(x) = \begin{cases} \pm\infty, & a \leqslant x < 0 \\ 0 & 0 \leqslant x < b \end{cases}$$

$z^0(x)$ is arbitrary, $y^0(x)$ does not belong to the C_1 class, consequently there is no strong minimum.

It follows from this example that in cases when theorem 3 is applicable, it is a stronger necessary condition than the Weierstrass condition.

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ON THE OPTIMAL MODE OF HORIZONTAL FLIGHT OF AN AIRPLANE

By

V. F. Krotov

In the last few years the practical requirements of technology present the investigator with an ever increasing number of problems pertaining to the regulation of the propulsive force of flying machines. Many of them were solved, concrete methods for solving problems of similar type were developed. However, despite the apparent variety, a single basic idea underlies all the methods. The problem of the minimum (maximum) of the function under consideration (fuel consumption range, time of flight ect) is examined within the framework of the classical problem of the calculus of variations, that is within the C_1 class of piecewise smooth functions. In view of this the problem of finding an extremum is refuced to the marginal problem for a system of differential equations (very often this problem is degenerated). In this manner it is apriori assumed that the extremum is piecewise smooth. However, as a result of the works (1), (2), (3) in the class of discontinuous functions (and the physics of the problem frequently require the examination of problems in this very class) it turns out that the typical extrema are those of a special kind (of the so-called type a) which, generally speaking, do not belong to the C_1 class and cannot be found within the gramework of the classical problem of the calculus of variations.

These extrema have the property of local independence of their elements from the neighboring elements and they also have the property of local maxima of position and slope, as a result of which properties the ultimate equations

of these extrema are written directly as necessary conditions of the extremum. The existence of such an extremum points to the absence of a strong minimum in the extremum of class C_1 if the latter does not coincide with it (see the previous article in the present collection).

It follows from what we have said that the methods of ordinary calculus of variations can not be considered to be general and satisfactory for the purpose of solution of problems of optimal regulations since: 1) they ignore the existence of an important class of solutions, 2) the solution, for certain types of functions, which is found by classical methods, such as Lagrange's method of multipliers, does not even give a relative minimum (maximum) due to the presence of extrema of the a type.

The same can be said about the new, intensively developing, directions in the calculus of variations: that is about Pontryagin's maximum principle and Bellman's theory of dynamics programming.

With all their advantages they suffer fully from the above mentioned drawbacks.

In the present work all these general considerations are illustrated by means of an actual example, which is also of significant interest on its own merits. The problem under consideration is that of optimal regulation of propulsive force for the horizontal flight of an airplane with a propulsion reactor, in order to achieve a maximum range. This problem has been investigated by Gibbs

/4/, and by M'yel' and Chikala /5/. According to the results obtained in /4/ and /5/, if we will assume a linear relationship between the propulsive force and the fuel consumption in the form

$$P = j\beta, \tag{1}$$

where P is the propulsive force, β is the fuel consumption, and j is constant effective velocity of flow; then the problem that has been formulated above is reduced to the finding of the extrema of a function having the form

$$\int_{m_k}^{m_c} [A(v; m) + v'B(v, m)]\, dm, \tag{2}$$

where m is the current value of the mass of the airplane, v is the airplane's velocity and $v' = \frac{dv}{dm}$.

The Euler equation for this type of function degenerates into the final equation

$$\frac{\partial A}{\partial v} - \frac{\partial B}{\partial m} = 0, \tag{3}$$

which in the general case does not pass through the given initial and final points (v_0, m_0), (v_k, m_k).

M'yel', by a brilliant application of Green's theorem, has formulated the desired solution, by showing that it consists of segments which satisfy the equation (3), in other words, the conditions for $\beta = \beta_{max}$ and $\beta = 0$ For the case of non-linearity of the relationship $P = P(\beta)$ the solution is not degenerative any more and it is possible to use Lagrange's method of multipliers for finding it.

In the present article the given problem is solved on the basis of another mathematical basic, namely with the help of the theory of non-continuous solutions of the calculus of variations which is presented in /1/, /2/, and /3/.

The dependency $P=P(\beta)$ is assumed to be arbitrary. Similar to /5/ we are looking for the dependency $v=v(m)$ which will assure maximum range at the given values of m_o, v_o and m_k, v_k. It is shown that the problem of finding a maximum reduces to finding the above relation for a function of two variables S(m,x, for each given value of $m \in (m_o m_k)$, where v and β are assumed to be independent. An optimal mode which is obtained in this manner turns out to be a degenerated one, and as a result it is independent of the position of the extremities (m_o,v_o) and (m_k,v_k).

If the equation $P=j\beta$ is of linear character (j=const t), the function S is independent of β and the solution $v=\bar{v}$ (m) coincides with that obtained in /4/ and /5/. If the equation $P=P(\beta)$ is of non-linear character, then the absolute maximum of the range is reached in the region of the so-called "pointwise propulsive force" (non-continuous tran) which is the above mentioned extremum of the type a.

This mode of operation consists of the following: the airplane flies from its initial position (m_o,v_o) with its motor shut off ($\beta=0$) or with its motor developing the maximum propulsion force (β =max), until it begins to follow a certain curve $v=\mathring{\bar{v}}(m)$ of the plane $(m,\bar{v})$. From then on the optimum region consists of a scheme when the flight with some optimal propulsive force P (β) during a certain time interval is followed by flight with the motor shut off (β =0,m=const), in which

case the motor is started again as soon as the velocity of the free flight falls below the value of $v=\bar{v}(m)$ and is again shut off as soon as that velocity is reached, this mode of operation continuing throughout the flight. In this case the frequency of starting and shutting off of the motor can be the maximum permissible one. The higher the frequency, the deeper the maximum. This mode of operation should be continued until the airplane begins to follow the line $\beta=0$ (m= const) or $\beta-\beta_{max}$, which line passes through the point (m_k, v_k) respectively if $v^0(m_k)>v_k$ or $\bar{v}^0(m_k)<v_k$.

Such an extremum obviously has nothing in common with the continuous, piecewise extremum that has been obtained by M'yel' and Chikala in /5/ as a solution of the marginal Lagrange problem.

The very fact that, as shown by the results of the previous article in this collection, such an extremum exists speaks loudly about the absence of an even relative maximum. In this manner, the solution of Chikala and M'yel' which has been obtained with the help of the methods of the classical calculus of variations is incorrect. A maximum is reached not upon the solution of the marginal problem of the Lagrange- Euler equations, but upon finding a non-continuous solution which is independent of its end points; the region of non-continuous propulsive force.

I am taking this opportunity to express my deepest appreciation to professor V. V. Dobronravov for a number of precious remarks which he has made in the editing of the manuscript.

Statement of the Problem

The equations of motion of an airplane flying in a straight horizontal path have the form:

$$\left.\begin{aligned} &m\dot{v}-P(H,v,\beta)+X(H,v,Y)=0;\\ &Y=mg;\\ &\beta=-\dot{m};\\ &\dot{x}-v=0, \end{aligned}\right\} \qquad (4)$$

where m is the mass of the airplane, g the acceleration of gravity, x the horizontal coordinate of the airplane, H the altitude of the horizontal flight, X any resistance, Y lifting force

$$\dot{v}=\frac{dv}{dt}, \quad \dot{m}=\frac{dm}{dt}.$$

We will state the following hypothesis about the forces that enter into the equations (4):

1) aerodynamic forces X and Y are independent of the plane's acceleration (we are not taking into account the aerodynamic lag); since the altitude is constant, X and Y are only dependent on the velocity; also, X is dependent on Y;

2) the propulsive force P depends on the velocity v and the per second consumption of fuel β and can be represented in the form

$$P(v,\beta)=f_1(v)f_2(\beta),$$

where f_1 (v) is the velocity characteristics of the propulsive force and is an arbitrary positive function, and f_2 (P) is the (fuel) consumption characteristic of the propulsive force, and is an ever increasing function. As a **rule** f_2 (0) 0. The physical significance of P(v,0) is resistive pressure:

$$P(v, 0) = -p(H)F_c,$$

where p (H) is a atmospheric pressure at the altitude H and F_c is the area of the nozzle's outlet. The dependency p (H) is of no special significance since h is constant.

(3) the acceleration of gravity is constant. In addition to this the equations (4) themselves were written on the assumption that the gravitational field of the earth is plane-parallel, that the effects of the earth's rotation can be neglected, and that the airplane can be considered as a particle.

We will formulate the boundary conditions. We will assume that at the instant when the optimal flight starts (t=0) the airplane has a mass m_0 and a velocity v_0. The beginning of the computation is chosen so that $x|_{t=0}=0$. At the finish of the flight we have $m=m_k$, $v=v_k$, $x=x_k$, while t_k and x_k are not fixed.

The problem consists in the finding of such a system of functions $x(t), v(t), m(t), \beta(t), Y(t)$, satisfying equalities (4) which will assure a maximum range x_k for a large number of sets $[x(t), v(t), m(t), \beta(t), Y(t)]$, which satisfy the system (4). We have 5 unknown functions and 4 equations of coupling, that is the system has one degree of freedom. From the last equation (4) we obtain an expression for the function that we wish to maximize

$$x_k = \int_0^{t_k} v\,dt = -\int_{t_k}^{0} v\,dt = \int_{m_k}^{m_0} \frac{v}{\beta}\,dm. \qquad (5)$$

Eliminating Y and dt from the first three equations (4) we will

obtain

$$\frac{dv}{dm} = -\frac{1}{m\beta}[P(v,\beta) - X(v,m)]. \tag{6}$$

We will consider an instantenous mass m as an independent variable in the integral (5). Since m(t) is a non-increasing function, then such a choice of an independent variable does not constrict the class of the lines of comparison, if the integral (5) is taken in the sense in which it was defined in /1/ and /3/.

The equation (6) assigns the function (m,v,v'). Substituting this expression into (5) we will obtain

$$x_k = \int_{m_k}^{m_0} F(m, v, v')\, dm. \tag{7}$$

In this manner the problem is reduced to the finding of a line v=v(m) on the surface (m,v) which line will give the absolute maximum of the function (7) in the class of lines which permit an unlimited number of vertical intervals, that is it is reduced to the problem investigated in /1/,/2/ and/3/.

We will now set the boundaries of the B region of physically permissible values of the function v=v(m). This boundary may consist of fragments of the following lines (fig. 1).

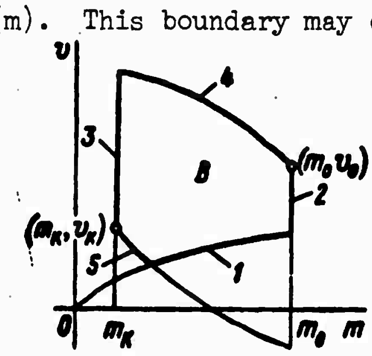

Fig. 1
The B-region of permissable values of the function v(m).

1) Line $mg-Y(v, a_{max})=0$, where a_{max} is the maximum permissible angle of attack, which represents by itself the lower boundary of permissible values of the function $v=v(m)$. 2) and 3) are vertical fragments of $m=m_o$ and $m=m_k$. 4) and 5) are the lines alongside which the flight under the influence of maximum propulsive force $\beta=\beta_{max}$ is taking place, the lines pass through the points (m_o, v_o) and (m_k, v_k) and they give the upper and lower limits, respectively, of the permissible values of $v(m)$. They are given by the equation (6) at $\beta=\beta_{max}$.

In addition to this it is necessary to impose the following limitations upon the variable of the function $v(m)$, namely:

$$\frac{dv}{dm} < \frac{1}{m\beta_{max}}[P(v, \beta_{max}) - X], \quad (8)$$

and also the condition of single valuedness of $v(m)$ in the respect that the line $v(m)$ can have not more than one common segment or point with any vertical straight line m=constant. The last condition follows from the above mention to the effect that the function $m(t)$ does not increase.

Finally the problem is stated in the following manner. In the class of functions $v=v(m)$ that pass through the points (m_k, v_k) and (m_o, v_o), which satisfy the condition (8) which belong to the B region and which contain an infinite number of discontinuties of the I kind it is required to find such a function $v=\bar{v}(m)$, for which the integral (7) taken in the case of the definition (8) from /3/ has an absolute maximum.

We will use the theory of non-continuous solutions of the problems of the calculus of variations which is developed in /1/, /2/ and /3/ for the solution of the given problem. According to the classification introduced in /1/ the function (7) which we are examining belongs to the type III. In order to prove this, we will find the characteristic function

$$W(m, v, \operatorname{sign} h) = \lim_{h \to \pm\infty} F(m, v, h)\frac{1}{h}. \tag{9}$$

Taking into consideration (5) and (7) we have

$$W(m, v, \pm 1)\, dv = \lim_{dm \to \pm 0} F\left(m, v, \frac{dv}{dm}\right) dm =$$

$$= -v\, dt(m, v, dm, dv)|_{dm \to \pm 0}. \tag{10}$$

Here $dt(m, v, dm, dv)$ is given by the first and third equations (4). Substituting the quantity $\beta = -\frac{dm}{dt}$, into the first equation (4) we will get

$$dt(m, v, dm, dv)|_{dm \to \pm 0} = -\frac{m\, dv}{X(m, v) - P(v, \beta)}\bigg|_{\beta \to \mp 0} \tag{11}$$

and consequently

$$W(m, v, \pm 1) = \frac{mv}{X - P(v, \beta)}\bigg|_{\beta \to \mp 0} = \frac{mv}{X - P(v, 0)}. \tag{12}$$

The quantity P(v,0) is always negative (reactive pressure) and because of this the limits of (12) exist for any given m and v.

The right side of (12) is independent of whether or not β approaches zero from the left or from the right, that is

$$W(m, v, +1) = W(m, v, -1); \tag{13}$$

everywhere in the B region of the m,v plane. Subsequently, according

to the classification of /1/ the function (7) belongs to the type III and can be written in the form

$$x_k = \int_{m_k}^{m_0} F(m, v, v')\,dm = \int_{m_k}^{m_0} S(m, v^0, z)\,dm + \Phi(m_0, v_0) - \\ - \Phi(m_k, v_k), \tag{14}$$

where

$$S = F(m, v^0, z) - W(m, v^0)\,z - \int_{C}^{v^0} W_m(m, \xi)\,d\xi; \tag{15}$$

$$\Phi(m, v) = \int_{C}^{v} W(m, \xi)\,d\xi; \tag{16}$$

$W_m(m,v)$ is a partical derivation of W in respect to m, and C is a derived constant.

In this case the sought function $v=v(m)$ can be totally undifferentiable. The function $v=v^0(m)$ is an independent piecewise smooth function which has the property of "null nearness" to $v(m)$ in the sense that

$$|v(m) - v^0(m)| < \varepsilon \qquad m \in (m_0, m_k), \tag{17}$$

where $\varepsilon > 0$ is any number.

The function z(m) has the same physical meaning as the derivative av/dm, but unlike the latter it is considered to be indepent of v.

In order to give more exact idea about the function $v(m)$, $v^0(m)$, $z(m)$

and of their interrelationship we will define the set of broken lines $v=v(m)$ in the (m,v) plane.

We will prescribe two piecewise smooth functions $v=v^0(m)$, $z=z(m)$ and we will break up the interval (m_k, m_o) into n fragment by the points m_i $(i=1,2, \ldots\ldots\ldots, n-1)$. We will construct the broken line $v=v_n(m)$ in the following manner (fig. 2);

$$v_n(m)=v^0(m_i)+z(m_i)(m-m_i); \text{ при } m_i<m<m_{i+1}. \quad (i=0,1,2,\ldots,n-1) \tag{18}$$

Then

$$v(m)=\lim_{|m_{i+1}-m_i|_{max}\to 0} v_n(m); \quad (i=0,1,\ldots,n-1) \tag{19}$$

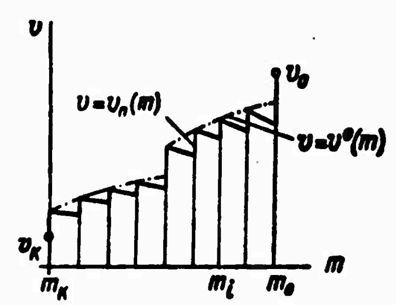

Fig. 2
Graph of functions $v_n(m)$.

From this definition and from the physical fact that for a constant altitude and with a shut off motor (β =), m=const) only a motion with a decreasing velocity is possible, it follows that the functions $v^0(m)$ and $z(m)$ should satisfy the following additional condition

$$\frac{dv^0}{dm}\geqslant z(m). \tag{20}$$

From (6) we have

$$z=-\frac{1}{m\beta}[P(v,\beta)-X(v,m)]. \tag{21}$$

This equation determines the relation between the quantities z and β .

Physically, the line $v=v_n(m)$ denotes the motion with periodic starting and shutting off of the motor. The sloped segments of the broken line correspond to the motion with the motor in operation, in which case the consumption β is given by the equation (21) with $z=z(m_i)$ $(i=0,1,\ldots\ldots n-1)$ and the vertical segments of the broken line correspond to the motion with the motor shut off. Each time the motor is turned on at the instant that the velocity of free flight at $m=m_i$ reaches the value $v^o(m_i)$ and is shut off when the mass reaches the value $m_i=1$. The total number of times when the motor is started is equal to n.

We will call this mode of operation, the mode of operation of non-continuous propulsion with a frequency n, with an average velocity $v^o(m)$ and with a propulsive force acting during the active intervals β (m) which is determined by (21).

If the amplified inequality (2) is satisfied, that is if $v^{o\prime}(m) \neq z(m)$ in respect to (m_k,m_o), then the function v(m), which is given by (19) represents by itself a region of non-continuous propulsion with infinite frequency.

Taking into consideration (21) we may consider function S to be dependent upon a pair of independent functions $v^o(m)$, z(m), or $v^o(m)$ β (m).

Taking into consideration (5), (12), (15) and (21) it is possible

to write $$S = \frac{v^0}{\beta} + \frac{m v^0}{X - P(v^0, 0)} \frac{1}{m\beta}[P(v^0,\beta) - X] - \int_c^{v^0} \left[\frac{m\xi}{X(\xi, m) - P(\xi, 0)}\right]_m d\xi$$

or, after abbreviations

where

$$S(m, v^0, \beta) = \frac{f_1(v^0)\, j(\beta)}{X(v^0, m) - P(v^0, 0)} - \int_c^{v^0} \left[\frac{m\xi}{X(\xi, m) - P(\xi, 0)}\right]_m d\xi, \tag{22}$$

$$j(\beta) = \frac{f_2(\beta) - f_2(0)}{\beta}. \tag{23}$$

In the case of an ideal liquid-fuel jet engine j is independent of β and it is the ideal relative velocity of flow.

For each fixed value of m the function S is dependent upon two independent variables v^0 and β. In accordance with /2/ and /3/, in order for the function (7), to have a maximum along the line v=v(m), it is necessary and sufficient that for each fixed m the function S should have a maximum along this line. In this manner, the problem of a maximum of a function is reduced to a problem of the maximum of a function of two variables $S(m, v^0, \beta)$.

The representing of such a propulsion force in the form $P(v, \beta) = f_1(v) f_2(\beta)$ permits us to give a quite simple method for the solution of the last problem. Namely the quantity $\beta = \beta$ which gives a maximum for any given m, $v \in B$, is not a function of m and v and it coincides with the point of the maximum of the function $j(\beta)$ for the interval $0 \leq \beta \leq \beta_{max}$. Figure 3 shows the variants of comsumptive characteristics of the propulsive force. We will draw from the

point $[0, f_1, 0]$ a family of rays which have at least one point in common with the curve f_2 ( ) in the interval $[0, \beta_{max}]$. It is easy to isolate from this family of rays γ, which produces the largest angle of inclination with the axis of abcissas. Abcissas $\bar{\beta}, \bar{\beta}_v$,........B_k of the common points of this ray with the curve f_2 (β) for interval $[0, \beta_{max}]$ will be the ones that will give the maximum j (β) that is the function S, which follows from (22). In this manner, if $P(v,\beta)$ can be represented in the form of the product f_1 (v) f_2 (β), then the optimum value = is independent of m and v.

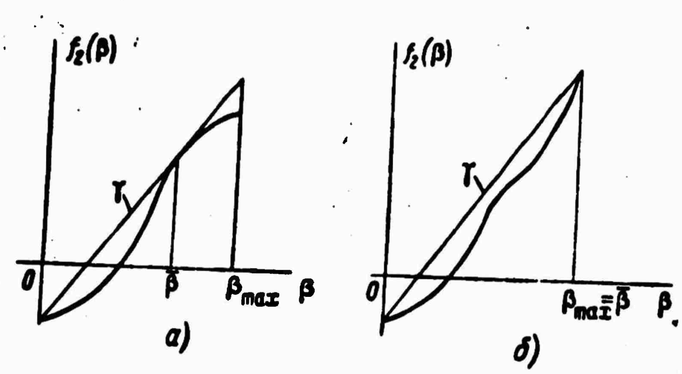

Fig. 3
Optimal consumption for different characteristics f_2 (β)

The optimal relationship $\overset{o}{v} = \bar{v}^o(m)$ is given by the condition of the absolute maximum of the function of one variable $S(m, j_{max}, v^0)$ in the B region for each fixed value of m. As a necessary condition the function $\bar{v}^o$ (m) should consist of the fragments of the boundary of the B region and from continuous fragments which satisfy the marginal equation

$$S_v = \left[\frac{f_1(v^0)\, j_{max}}{X(v^0,m) - P(v^0,0)}\right]_{v^0} - \left[\frac{m v^0}{X - P(v^0,0)}\right]_m = 0 \qquad (24)$$

and of fragments connected by vertical lines at the points $m = \mu_j$ $(j = 1, 2, \ldots, r)$, for which the condition

$$S[\mu_j, j_{max}, v_1(\mu_j)] = S[\mu_j, j_{max}, v_2(\mu_j)], \qquad (25)$$

is satisfied. Here $v_1(m)$ and $v_2(m)$ are two solutions of the equation (21).

The equation (24) coincides with the equation $\omega=0$ of the so-called "special curve" from /5/ that is with the degenerated Euler equation for the case of linear dependence of the propulsive force upon the (fuel) consumption, if we should assume $P(v, 0)=0, f_1(v)\equiv 1$ and if the effective velocity of flow is constant, that is $f_{max}=v_r=const.$. Here the first two conditions represent by themselves the simplifying assumptions that were made in /5/. This fact makes it possible to apply the results of /5/ as a valuation of the function $v=v^o(m)$, namely: all the properties of the maximal function $v=v(m)$ which are examined in detail in /5/ exist for those functions; since 1) the equation (24) has two solutions, supersonic $v^0=\bar{v}_1^0(m)>a$ and subsonic $v^0=v_2^0(m)<a$, where $a(H)$ is the speed of sound at the altitude H. Along both solutions $\frac{dv^0}{dm}>0$, /that is the motion takes place with decreasing velocity, in which case $\bar{v}^o_2(m)$ always passes through the origin of coordinates, 2) the optimal flight $v^o=\bar{v}^o(m)$ must take place either along one of the solutions of (24) or first along the line $v_1(m)$ till the point $m=\mu$, which satisfies the condition (25) which coincides with (50) from /5/, when free flight along the vertical $m=\mu$ until it emerges into $v_2(m)$ and further along the line $v_2(m)$.

The emergence from the point (m_o, v_o) into the solution of (24) and from the latter into the point (m_k, v_k) should be taking place along the corresponding interval of the boundary of the B region, that is either at $\beta=0$ or at $\beta=\beta_{max}$, depending on the position of points (m_o, v_o) and (m_k, v_k) respectively, underneath

the line $v^0=\bar{v}^0(m)$ or on top of it (fig. 4).

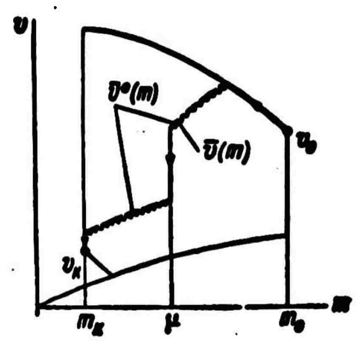

Fig. 4
Optimal region of non-continuous propulsion $\check{v}$ (m).

In accordance with the results obtained in /1/, /2/ and /3/ the whole optimal function has as its necessary and sufficient condition for the absolute maximum of the function of one variable $S(m, j_{max}, v^0)$ in the B region for each fixed value of m.

In this manner we have found the maximal values of the consumption β for the sloping fragments [and subsequently also for z(m)] which is independent of m and v and correspond to the maximum value of $j(\beta)$, and we have also found the maximal function $v^0=\bar{v}^0(m)$, which coincides with the maximum velocity $\bar{v}(m)$, if the velocity of flow is considered to be constant and equal to j_{max}.

In order to explain the character of the extremum, we are only left with the problem of proving the inequality (20). As has been shown above $\frac{dv^0}{dm}>0$. C along both extremal branches. On the other hand, according to (21) $z(m)\leqslant 0$ in all cases when

$$P(v, \beta) > X(v, m). \qquad (26)$$

The last equation almost applies; that is as a rule propulsive force which corresponds to a maximal j is such that it is capable of overcoming head-on resistance, but in this case the condition (21) is even better satisfied. The greater the head-on resistance X(v,m), the larger the v and m, because of this, even if the condition (26) is not satisfied, this only happens for the supersonic branch $v_1(m)$. But for this branch $\frac{dv^\circ}{dm}$ also reaches the highest value (see /5/ page (133) and because of this (21) is satisfied even in this case, even if (26) is not satisfied.

Despite this fact we will also examine a case for which (26) is not satisfied, since all these cases are permissible and, besides, are of theoretical interest.

In the line $v^\circ=\bar{v}^\circ(m)$ is also included the $\beta=\beta_{max}$ fragment of the boundary of the B region, for which the equation (21) is not satisfied (excluding the case where $\bar{\beta}=\beta_{max}$) and according to /2/ [see formulas (11)] we should assume $\bar{\beta}=\beta_{max}$.

We will now examine extremals of the form $v=\bar{v}(m)$ for various characteristics particular cases.

I. There exists a finite number of values $\bar{\beta}_i$ $(i=1, 2, \ldots, k)$, which satisfy the condition $j(\bar{\beta}_i)=j_{max}$ [the finite number of common points that ray γ has with the characteristic curve f_2 (β) for $(0,\beta_{max})$]. All the $\bar{\beta}_i$ satisfy the inequality (21). The quantity $\beta^\circ(m)$ is given by the equation

$$\frac{d\bar{v}^0(m)}{dm} = -\frac{1}{m\beta}\left[P(\bar{v}^0, \beta^0) - X(\bar{v}^0, m)\right]. \tag{27}$$

which changes continuously with changing m. But since on the other hand any of the optimal values $\beta = \bar{\beta}_i = \text{const}$, then $z(m) \neq \frac{d\bar{v}}{dm}$. Consequently the optimal region $v = \bar{v}(m)$ represents the region of non-continuous propulsive force with infinite frequency that has been described above. This means that the point situated in the (m,v) plane which point represents the airplane (see Fig. 4) should move from the position (m_0, v_0) along the boundary of the B region (region of $\beta = 0$ cr $\beta = \beta$ max) up to such time, until it will emerge onto the line of null nearness $v^0 = \bar{v}^0(m)$.

From then on the operation with non-continuous propulsion with the maximum permissible frequency begins, in which case the (fuel) consumption for the active intervals should be equal to one of the values β_i (any value), and the average velocity will be the obtained function $\bar{v}^0(m)$. This mode of operation should continue up to the time until the representative points emerges on the boundary. Upon this the motion should be taking place along the boundary until the final position (m_k, v_k) is reached.

II. There exists a finite number of values $\bar{\beta}_i \ (i = 1, 2, \ldots, k)$, in which case all the β_i do not satisfy the inequality (21). In this case the optimal physically permissible function z(m) is represented by

$$z(m) = v^{0'}(m). \tag{28}$$

This means that the sought optimal function $v = \bar{v}(m)$ should be piecewise smooth. In this manner, in the given case the finding of the

extremum has been reduced to the classical problem of the calculus of variations in the class of piecewise smooth lines. It can be solved by the method, which is presented in /5/ for the case of non-linear relationships of $P(\beta)$, however, with one substantial correction. As boundary conditions the conditions of transversality of the extremum to the boundary of the B region should be used, and not the condition that the extremum must be passing through the points (m_o, v_o) and (m_k, v_k). A more detailed analysis of this case will not be given since in accordance with what we said above this case can only occur as an exception.

III. There exists a finite number of values $\bar{\beta}_i$ $(i = 1, 2, \ldots, k)$ some of which satisfy the inequality (21) and some do not. The sought optimum is the mode of operation with non-continuous propulsion similar to the case I. The fuel consumption during the active intervals should be equal to any of the values β_i, which satisfy (21).

IV. There exists a continuous continuum of values $0 < \bar{\beta}_0 < \bar{\beta} < \bar{\beta}_k < \beta_{max}$, for which $j(\beta) = j_{max}$, that is the ray γ has an interval in common with the propulsion characteristic $f_2(\beta)$ (fig. 5). A part of the interval satisfies the inequality (21), another part may not satisfy it. The particular case of $\bar{\beta}_o = 0, \beta_{max} = \beta_K$ is the linear characteristic of the propulsion. In the given case there exists an infinite number of optimal solutions. They are represented by the regions of non-continuous propulsion with infinite

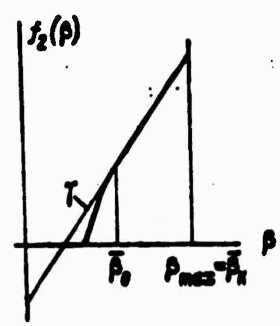

Fig. 5
Case in which the ray γ has a common interval with the characteristic $f_2(\beta)$

frequency and with a common line of null nearness, similar to the case I but differing from it im the fact that any relationship $\beta(m) \in [\bar{\beta}_0, \bar{\beta}_k]$ can exist on the aciive intervals as long as it satisfies inequality (21), and included in the permissible relationships is also $\beta = \beta^{o}(m)$ which turns the inequality (21) into an equality. In the last case the mode of operation with non-continuous propulsion degenerates into a mode of operation with continuous guidance, which has been obtained by M'yel' in /5/ for the case of linear dependency of P (β), and the maximal function $v=\bar{v}(m)$ coincides with the line of null nearness $v=\bar{v}^{o}(m)$.

The physical meaning of the non-uniqueness of the solution in the given case consists of the fact that for the range of flight only the average velocity $v=v^{o}(m)$ exists and all the local divergences from it are meaningless.

However, if we will take into consideration the fact that the practical frequency of shutting off and starting is finite, them from all the infinite number of maxima there stands out a unique one, that one for which a strict maximum of range is reached, namely, the continuous mode of operation with regulation of the propulsion which has been presented above.

The mathematical sense of the infinite numbers of solutions consists in that j is constant for the interval $[\bar{\beta}_o, \bar{\beta}_k]$, and as a result the function $S(m,j,v^{o})$ is independent of but happens to be a function of one variable v^{o} for each fixed value of m.

Conclusions

The paper has examined the application of the theory of non-continuous solution of problems of the calculus of variations, which has been developed in /1/, /2/ and /3/ to be the problem of finding of optimal regulation of the work of a motor for the condition of horizontal flight of an airplane.

A method was developed for the direct determination of optimal modes of propulsion for different relationships between the propulsive force and the fuel consumption.

For the case of the linear dependency of the propulsive force on the consumption there was obtained maximal regions which coincide with the results obtained by M'yel' /5/. We have obtained new necessary and sufficient conditions for the absolute (and not relative as in /5/) maximum of range, which are much more convenient in computations, in the form of the condition of maximum of the function of one variable $S(m,j,v^{o})$ for each given fixed value of m.

We have shown the incorrectness of the results obtained by Chikala in /5/ by means of the classical methods of the calculus of variations for the case of non-linearity of the dependency $P(\beta)$. It was shown that the maximum of the range is reached within the optimal regions which are basically different from the ordinary continuous optima and which cannot be obtained by the methods of classical calculus of variations. These optimals are represented by the operation with non-continuous propulsion, which consist of

of a combination of the maximum permissible number of flight intervals with the motor shut off and of intervals of some predetermined optimal propulsion and with a maximal initial velocity $\bar{v}^{o}(m)$.

From the technical point of view, the basic result consists of the discovery of new optimal regions of non-cnntinuous propulsion in the problem of maximized programming and in the investigation and setting up on such regions within the problem of maximum range of the case of horizontal flight of an airplane.

From the mathematical point of view, the basic result consists in the showing by means of a practical example the inadequacy, and in many cases of the incorrectness of the applications of the classical methods of the calculus of variations to the problem of optimal regulations: and in the case of horizontal flight the basic result consists in finding the necessary and sufficient conditions for the absolute maximum of range and in the developing of a quite simple method for the direct findings of extremals for arbitrary dependencies of the propulsive force upon the consumption and velocity.

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ON THE FORCED VIBRATIONS OF A RECIPROCATING HYDRAULICAL SERVO WITHOUT FEEDBACK

By

Yu. Ye. Zakharov and V. N. Baranov

Hydraulical reciprocating servos without feedback and with valve controls are at the present time widely used in machine tools, turbines, steering machinery etc. In service these servos are required to reproduce periodic incoming signals for various forms of pressure on the piston.

The types of pressure most often encountered are:

1) a force of uniform value and direction
2) elastic force (for instance, if the piston is spring-loaded)
3) inertial forces (when the mass of the loading medium cannot be neglected).

The motion of the piston of a hydraulic servo for typical loads with a constant output signal (constant displacement of the valve) is given a quite detailed description in /1/. The motion of the piston of a hydraulic servo without taking into consideration the mass of the loading medium was investigated in /2/. Forced vibration of the piston of a hydraulic servo taking into consideration the mass of the loading medium, but not the elastic force were examined in /3/ and /4/.

In this article there are examined the periodic motions of the piston of a hydraulic servo taking into consideration three basic types of piston loadings for the case of an arbitrary input signal.

A basic schematic drawing of a reciprocating hydraulic servo is given in Fig. 1.

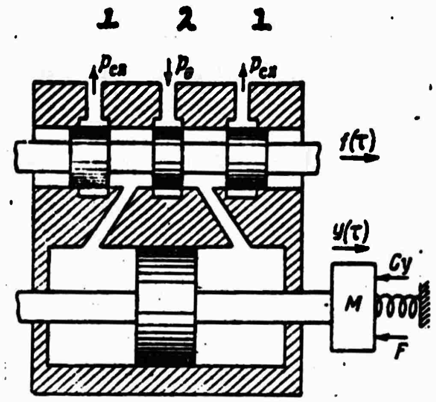


Figure 1

Schematic drawing of a reciprocating hydraulic servo without feed-back. 1. discharge; 2. supply.

In the finding of the solution of the equation of motion of the piston we are making the usual assumptions:

1) the working fluid is incompressible and the walls of the cylinder are infinitely stiff,

2) the temperature and viscosity of the working fluid and the pressures fed into the servo are constant

3) there is no leakage from the regions of the servo

4) viscous friction between the cylinder and the walls is negligibly small

5) the feed pipes are non-deformable and short, so that the pressure losses in these pipes are negligiblely small

6) the regulating valve and the piston are symmetrically situated (in respect to one another ? trans),

7) the force of (dry) friction acting on the piston is constant.

With the above assumptions the equation of motion of the piston of a hydraulic servo will have the form

$$M\ddot{y}+Cy+F+\left[\frac{\gamma S^3\dot{y}^2}{g\mu^2 f^2(t)}-S(p_0-p_{ca})+R\right]\operatorname{sign} f(t)=0, \quad (1)$$

where y is the coordinate of the piston

M - mass of the loading medium (which includes the mass of the piston itself and of the fluid which sticks to it)

C - the proportionality coefficient of the tensile force of the load (stiffness of the load)

A-1 S - effective area of the piston

- specific weight of the oil

g - acceleration of gravity

- coefficient of consumption in the apertures of the valve

f(t) - area of the open valve aperture

R - force of dry friction acting on the piston

F - constant loading force

P_o - supply pressure

P - discharge pressure

It is convenient to make the analysis of motion by using dimensionless variables, which are given by the formulas

$\tau = \omega t$ - dimensionless time

$\delta = \frac{y}{x_{max}}$ - dimensionless piston displacement

$\varphi(\tau) = \frac{f(t)}{f_{max}}$ - dimensionless area of the opening of the valve apertures

where ω, x_{max} - angular frequency and amplitude (respectively) of the first (basic harmonic of the periodic input signal

f_{max} - maximum opening of the valve aperture.

When the dimensionless variables are used the equation of motion of the piston of the hydraulic servo is in the form

$$\varepsilon\frac{d^2\delta}{d\tau^2}+\lambda_0^2\delta+\alpha+\left[\frac{\left(\frac{d\delta}{d\tau}\right)^2}{k^2\varphi^2(\tau)}-1\right]\operatorname{sign}\varphi=0, \tag{2}$$

where $\varepsilon=\frac{M\omega^2x_{max}}{S(p_0-p_{ca})-R_0}$ — is a dimensionless coefficient which can be considered as the ratio of the maximal inertia force for the case of vibration with the frequency ω, to the useful effort exerted on the piston.

$\lambda^2=\frac{C_0x_{max}}{S(p_0-p_{ca})-R_0}$ - dimensionless coefficient which can be considered as the ratio of the maximal linear tensile force to the useful effort exerted on the piston.

$k^2=\frac{g\mu^2f_{max}^2[S(p_0-p_{ca})-R_0]}{S^3\gamma x_{max}^2\omega^2}$ — is a dimensionless parameter which takes into consideration the dimensionless of the hydraulic servo.

$\alpha=\frac{F}{S(p_0-p_{ca})-R_0}$ — is a ratio of the constant loading force to the useful effort exerted on the piston.

By multiplying all the members of the equation (2) by sign $f(\tau)$ and by solving for $d\delta'/d\tau$ we will obtain

$$\frac{d\delta}{d\tau}=k\varphi(\tau)\sqrt{1-\left(\alpha+\lambda^2\delta+\varepsilon\frac{d^2\delta}{d\tau^2}\right)\operatorname{sign}\varphi}. \tag{3}$$

Introducing a new variable z according to the formula

$$1-(\alpha+\lambda^2\delta)\operatorname{sign}\varphi=\lambda^2z, \tag{4}$$

we will have

$$\frac{dz}{d\tau}=-k\varphi(\tau)\sqrt{\lambda^2z+\varepsilon\frac{d^2z}{d\tau^2}\operatorname{sign}\varphi}. \tag{5}$$

In hydraulic servos the parameter $\varepsilon \ll 1$. Actually, in order for the servo to work as a motor it is necessary to meet the following condition

$$1-\left(a+\lambda^2 z_0+\varepsilon\frac{d^2 z}{d\tau^2}\right)\operatorname{sign}\varphi>0;$$

that is it is necessary to ascertain that the useful effort exerted on the piston should be greater then the sum of all the resistances (load forces).

In order to solve the non-linear equation (5) we will use the method of small parameter using the parameter ε for this purpose.

We are looking for a solution in the form of a series arranged according to the powers of ε with coefficients subject to the definition

$$z=z_0+\varepsilon z_1+\varepsilon^2 z_2+\ldots$$

Substituting this expression into equation (5) and expanding the expression beneath the radical sign in a series according to the powers of ε, and by collecting the coefficients of terms with the same magnitudes of power of ε we will obtain the following equations for the determination of the approximate valves of z_i:

$$\frac{dz_0}{d\tau}=-k\lambda\varphi(\tau)\sqrt{z_0}\operatorname{sign}\varphi; \qquad (6)$$

$$\frac{dz_1}{d\tau}=-\frac{k\lambda\varphi(\tau)}{2\sqrt{z_0}}\left(z_1+\frac{1}{\lambda^2}\frac{d^2 z_0}{d\tau^2}\right)\operatorname{sign}\varphi; \qquad (7)$$

$$\frac{dz_2}{d\tau}=-\frac{k\lambda\varphi(\tau)}{2\sqrt{z_0}}\left(z_2+\frac{1}{\lambda^2}\frac{d^2 z}{d\tau^2}-\frac{1}{4z_0\lambda^2}z_1-\frac{1}{4\lambda^4 z_0}\frac{d^2 z_0}{d\tau^2}\right)\operatorname{sign}\varphi. \qquad (8)$$

In our further discussion we will limit ourselves to the first two approximations. Separating variables in equation (6) we get

$$\frac{dz_0}{\sqrt{z_0}} = -k\lambda\psi(\tau)\,d\tau; \qquad \psi(\tau) = \varphi(\tau)\,\mathrm{sign}\,\varphi.$$

Now by integration we find the first approximation

$$\sqrt{z_0} = -\frac{k\lambda}{2}\left(\int\psi(\tau)\,d\tau + C_0\right);$$
$$z_0 = \frac{k^2\lambda^2}{4}\left[\int\psi(\tau)\,d\tau + C_0\right]^2. \tag{9}$$

where C is an arbitrary constant.

As a result

$$\frac{dz_0}{d\tau} = \frac{k^2\lambda^2}{2}\psi(\tau)\left[\int\psi(\tau)\,d\tau + C_0\right];$$
$$\frac{d^2z_0}{d\tau^2} = \frac{k^2\lambda^2}{2}\{\psi'(\tau)[\psi(\tau)\,d\tau + C_0] + \psi^2(\tau)\}.$$

Substituting these expressions into equation (7) we will get a linear equation of the first order in respect to z_1

$$\frac{dz_1}{d\tau} - \frac{\psi(\tau)}{\int\psi(\tau)\,d\tau + C_0}z_1 - \frac{k^2}{2}\left[\psi'(\tau)\psi(\tau) + \frac{\psi^3(\tau)}{\int\psi(\tau)\,d\tau + C_0}\right] = 0.$$

The solution of this equation has the form

$$z_1 = e^{\int\frac{\psi(\tau)}{\int\psi(\tau)\,d\tau + C_0}d\tau}\left\{C_1 + \frac{k^2}{2}\int\left[\psi'(\tau)\psi(\tau) + \right.\right.$$
$$\left.\left. + \frac{\psi^3(\tau)}{\int\psi(\tau)\,d\tau + C_0}\right]e^{-\int\frac{\psi(\tau)}{\int\psi(\tau)\,d\tau + C_0}d\tau}\right\}, \tag{10}$$

where C_1 is an arbitrary constant.

We will now transform the expression for the second approximation keeping in mind that

$$\int \frac{\psi(\tau)\,d\tau}{\int \psi(\tau)\,d\tau + C_0} = \ln\left[\int \psi(\tau)\,d\tau + C_0\right];$$
$$e^{\ln\left[\int \psi(\tau)\,d\tau + C_0\right]} = \int \psi(\tau)\,d\tau + C_0.$$

Finally we get

$$z_1 = C_1\left[\int \psi(\tau)\,d\tau + C_0\right] - \frac{k^2}{2}\psi^2(\tau) + \tag{11}$$
$$+ \frac{3k^2}{2}\left[\int \psi(\tau)\,d\tau + C_0\right]\int \frac{\psi'(\tau)\psi(\tau)}{\int \psi(\tau)\,d\tau + C_0}\,d\tau.$$

Having found the variables z_1 we will turn to the finding of the dimensionless piston displacement δ .

From the formula (4) we have

$$\delta = \left(\frac{1}{\lambda^2} - z\right)\operatorname{sign}\varphi(\delta) - \alpha.$$

Keeping in mind that $\delta = \delta_0 + \varepsilon\delta_1 + \ldots$ and collecting the coefficients of equal values of powers of ε , we get

$$\delta_0 = \left(\frac{1}{\lambda^2} - z_0\right)\operatorname{sign}\varphi(\tau) - \alpha; \tag{12}$$
$$\delta_1 = -z_1\operatorname{sign}\varphi(\tau). \tag{13}$$

In connection with the fact that the equations for finding approximate values are equations of the first order, each approximation contains only one arbitrary constant. An exact solution of the original equation, if such can be found, should contain two arbitrary constants. Due to this fact the approximate solution that has been obtained, is not applicable to the analysis of self-induced vibrations of the piston of a hydraulic servo, however, in the analysis of forced vibrations, when self-induced vibrations are not of interest, the absence of one arbitrary constant is unimportant. The one arbitrary constant that does exist serves the purpose of providing a continuity of

solution at the boundary points.

As an example we will examine a sinusoidal input signal $f(\tau) = \sin\tau$ during one period of vibration for this case $\psi(\tau) = \sin\tau \operatorname{sign}\tau$ and as a result

$$\psi(\tau)=\begin{cases} \sin\tau; & 0\leqslant\tau\leqslant\pi; \\ -\sin\tau; & \pi\leqslant\tau\leqslant 2\pi; \end{cases} \qquad \psi'(\tau)=\begin{cases} \cos\tau; & 0\leqslant\tau\leqslant\pi; \\ -\cos\tau; & \pi\leqslant\tau\leqslant 2\pi; \end{cases}$$

$$\int\psi(\tau)\,d\tau=\begin{cases} -\cos\tau; & 0\leqslant\tau\leqslant\pi; \\ \cos\tau; & \pi\leqslant\tau\leqslant 2\pi. \end{cases}$$

The graphs of functions $\psi'(\tau)$, $\psi(\tau)$, $\int\psi(\tau)d\tau$ are given in fig. 2.

According to formula (9) we get the first approximation in the form

$$\sqrt{z_0}=\begin{cases} \frac{k\lambda}{2}\cos\tau+C_{01} & 0\leqslant\tau\leqslant\pi; \\ -\frac{k\lambda}{2}\cos\tau+C_{02} & \pi\leqslant\tau\leqslant 2\pi. \end{cases}$$

From which it follows

$$z_0=\begin{cases} \frac{k^2\lambda^2}{4}\cos^2\tau+k\lambda C_{01}\cos\tau+C_{01}^2 & 0\leqslant\tau\leqslant\pi; \\ \frac{k^2\lambda^2}{4}\cos^2\delta-k\lambda C_{02}\cos\tau+C_{02}^2 & \pi\leqslant\tau\leqslant 2\pi. \end{cases}$$

The second approximation is obtained from formula (11)

$$z_1=\begin{cases} -C_{11}(\cos\tau-C_{01})-\frac{k^2}{2}\sin^2\tau-\frac{3k^2}{2}(\cos\tau- \\ \quad -C_{01})[\cos\tau+C_{01}\ln|\cos\tau-C_{01}|]; \\ \quad 0\leqslant\tau\leqslant\pi \\ C_{12}(\cos\tau+C_{02})-\frac{k^2}{2}\sin^2\tau-\frac{3k^2}{2}(\cos\tau+ \\ \quad +C_{02})[\cos\tau-C_{02}\ln|\cos\tau+C_{02}|] \\ \quad \pi\leqslant\tau\leqslant 2\pi \end{cases}$$

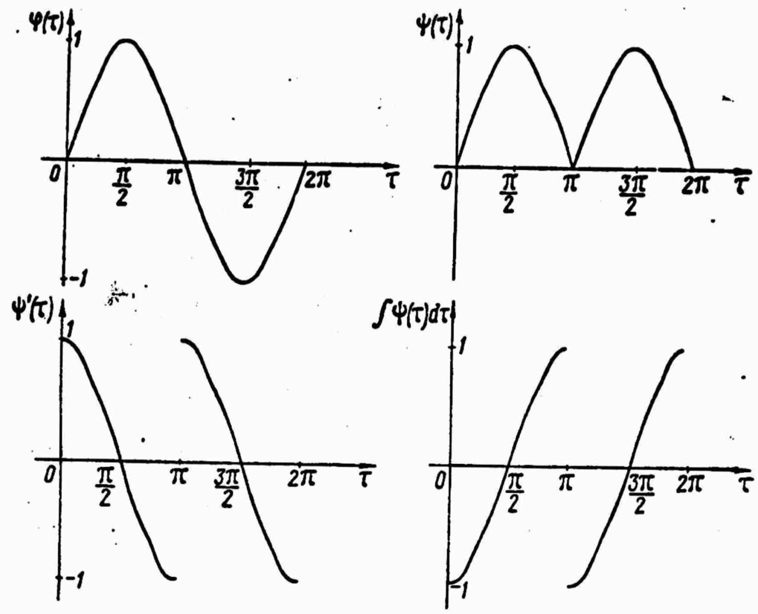


Figure 2

Graph of functions $\varphi(\tau) = \sin\tau,\ \psi(\tau) = \sin\tau \operatorname{sign} \sin\tau,$
$\psi'(\tau) = \cos\tau \operatorname{sign} \sin\tau,\ \int \psi(\tau)\,d\tau = -\cos\tau \operatorname{sign}\sin\tau.$

From formula (12) we define the dimensionless (piston) displacement δ_0 as follows:

$$\delta_0 = \begin{cases} \left(\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}\cos^2\tau - k\lambda C_{01}\cos\tau - C_{01}^2\right) - \alpha & 0 \leqslant \tau \leqslant \pi; \\ -\left(\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}\cos^2\tau + k\lambda C_{02}\cos\tau - C_{02}^2\right) - \alpha & \pi \leqslant \tau \leqslant 2\pi. \end{cases} \tag{14}$$

The arbitrary constants C_{01} and C_{02} are determined from the condition of continuity at the boundary as follows:

$$\delta_{01}(0) = \delta_{02}(2\pi); \quad \delta_{01}(\pi) = \delta_{02}(\pi).$$

Substituting the appropriate values of τ into the formula (14) we get

$$\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4} - k\lambda C_{01} - C_{01}^2 = -\frac{1}{\lambda^2} + \frac{k^2\lambda^2}{4} - k\lambda C_{02} + C_{02}^2;$$
$$\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4} + k\lambda C_{01} - C_{01}^2 = -\frac{1}{\lambda^2} + \frac{k^2\lambda^2}{4} + k\lambda C_{02} + C_{02}^2.$$

By collecting term by term and subtracting, we get a system of equations for the determination of the arbitrary constants C_{01} and C_{02}

$$C_{01}^2 + C_{02}^2 = \frac{2}{\lambda^2} - \frac{k^2\lambda^2}{2}; \quad k\lambda (C_{01} - C_{02}) = 0,$$

from where

$$C_{01} = C_{02} = C_0 = \sqrt{\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}}. \tag{15}$$

As a result we finally have

$$\delta_0 = \begin{cases} -\frac{k^2\lambda^2}{4}\cos^2\tau - k\lambda\sqrt{\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}}\cos\tau + \frac{k^2\lambda^2}{4} - \alpha & 0 \leqslant \tau \leqslant \pi; \\ \frac{k^2\lambda^2}{4}\cos^2\tau - k\lambda\sqrt{\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}}\cos\tau - \frac{k^2\lambda^2}{4} - \alpha & \pi \leqslant \tau \leqslant 2\pi. \end{cases}$$

These two formulas can be put together as follows

$$\delta_0 = \frac{k^2\lambda^2}{8}(1 - \cos 2\tau)\,\mathrm{sign}\sin\tau - k\lambda\sqrt{\frac{1}{\lambda^2} - \frac{k^2\lambda^2}{4}}\cos\tau - \alpha \qquad 0 \leqslant \tau \leqslant 2\pi. \tag{16}$$

We get the dimensionless displacement δ, from formula (13) and follows

$$\delta_1 = \begin{cases} C_{11}(\cos\tau - C_0) + \frac{k^2}{2}\sin^2\tau + \\ + \frac{3k^2}{2}(\cos\tau - C_0)[\cos\tau + C_0 \ln|\cos\tau - C_0|] & 0 \leqslant \tau \leqslant \pi; \\ C_{12}(\cos\tau + C_0) - \frac{k^2}{2}\sin^2\tau - \\ - \frac{3k^2}{2}(\cos\tau + C_0)[\cos\tau - C_0 \ln|\cos\tau + C_0|] & \pi \leqslant \tau \leqslant 2\pi. \end{cases}$$

The arbitrary constants C_{11} and C_{12} are found from the conditions of continuity at the boundary as follows:

$$\delta_{11}(0) = \delta_{12}(2\pi); \quad \delta_{11}(\pi) = \delta_{12}(\pi),$$

which gives

$$C_{11}(1 - C_0) + \frac{3k^2}{2}(1 - C_0)[1 + C_0 \ln|1 - C_0|] = C_{12}(1 + C_0) - \\ - \frac{3k^2}{2}(1 + C_0)[1 - C_0 \ln|1 + C_0|];$$

$$-C_{11}(1+C_0)-\frac{3k^2}{2}(1+C_0)[-1+C_0\ln|1+C_0|]=$$
$$=-C_{12}(1-C_0)-\frac{3k^2}{2}(1-C_0)[1+C_0\ln|1-C_0|].$$

By solving this system of equations we obtain the value of the arbitrary constant C_1:

$$C_1=C_{11}=C_{12}=\frac{3k^2}{4C_0}\left[2+C_0\ln\left|\frac{1-C_0}{1+C_0}\right|-C_0^2\ln|1-C_0^2|\right]. \quad (17)$$

Taking into consideration the above proven fact of inequality of the arbitrary constant, and by collecting terms we get

$$\delta_1=\left[C_1-\frac{3k^2}{2}C_0+\frac{3k^2}{2}C_0\ln|\cos\tau-C_0\operatorname{sign}\sin\tau|\right]\cos\tau+$$
$$+\left[k^2-C_1C_0+\frac{k^2}{2}\cos 2\tau-\frac{3k^2}{2}C_0^2\ln|\cos\tau-C_0\operatorname{sign}\sin\tau|\right]\operatorname{sign}\sin\tau, \quad (18)$$
$$0\leqslant\tau\leqslant 2\pi,$$

where the arbitrary constants C_0 and C_1 are given by formulas (15) and (17) respectively.

We will now find the permissible limits within which the arbitrary constant C_0 may vary. The condition for which the hydraulic servo will operate as a motor, can be formulated as follows if we neglect the terms which contain the small parameter ε

$$1-(\alpha+\lambda^2\delta_0)\operatorname{sign}\sin\tau>0$$

or in periodic order

$$1-\alpha-\lambda^2\delta_{0max}>0 \quad 0\leqslant\tau\leqslant\pi;$$
$$1+\alpha+\lambda^2\delta_{0max}>0 \quad \pi\leqslant\tau\leqslant 2\pi.$$

We will assume that $C_0=0$, then

$$\delta_0=\frac{k^2\lambda^2}{8}(1-\cos 2\tau)-\alpha;$$
$$0\leqslant\tau\leqslant\pi$$
$$\delta_{0max}=\frac{k^2\lambda^2}{4}-\alpha=\frac{1}{\lambda^2}-\alpha.$$

But if we were to substitute this value of $\delta_{0\,max}$, then $1-\alpha-\lambda^2\delta_{0max}=0$, which is impossible.

Therefore, $C_0 > 0$. This condition defines the limitation which is imposed upon the parameter λ^2 as follows:

$$\frac{1}{\lambda^2}-\frac{k^2\lambda^2}{4}>0 \quad \lambda^2<\frac{2}{k}.$$

Fig. 3 is a graph of a hydraulic servo piston which has been constructed according to formulas (16) and (18) and for the conditions $\lambda^2=0{,}5;\ k^2=8;\ \varepsilon=0{,}1;$ In this case $C_0=1$, $C_1=3.72$. Calculations that have been made show that $\ddot{\delta}=3.4$ for $\tau=\pi$, and $\ddot{\delta}=0.4$ for $\tau=2\pi$. As a result, the servo acts as a motor. In actuality for $\tau=\pi$ ($\delta=\delta_{max}=1.5$) we have

$$1-\alpha+\lambda^2\delta-\varepsilon\ddot{\delta}-\ldots=1-0{,}5-0{,}75+0{,}34-\ldots>0,$$

in this case the terms that have been disregarded do not exceed $\varepsilon=0.1$.

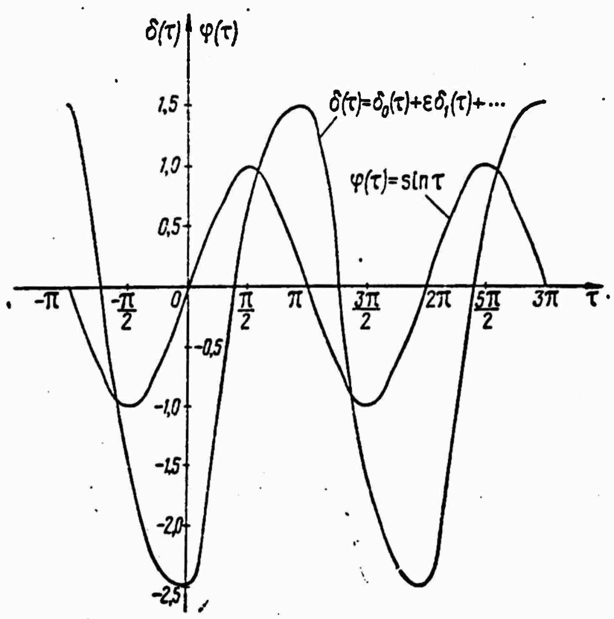


Figure 3
Graph of the displacements of the piston of a hydraulic servo.

For $\tau = 2\pi$ ($\delta = \delta_{max\,2} = 2.5$) we have

$$\tau = 2\pi \quad (\delta = \delta_{max2} = -2.5)$$
$$1 + \alpha + \lambda^2\delta + \varepsilon\ddot{\delta} \ldots = 1 + 0{,}5 - 1{,}25 + 0{,}04 \ldots > 0.$$

As can be seen from graph 3, the piston of a hydraulic servo without feedback does not follow the amplitude of vibrations of the valve, the amplitude of the piston's vibrations is determined basically by the parameters of the servo mainly by the rigidity and the mass of the loading medium. For practical computations the amplitude of the piston's vibrations can be approximately determined from the following considerations: with a degree of accuracy which is sufficient for a trial solution it is possible to postulate that for a sinusoidal motion of the valve the piston vibrates according to the cosine law

$$\delta = \delta_{max} \cos\tau; \quad \frac{d^2\delta}{d\tau^2} = -\delta_{max} \cos\tau.$$

At the instant when the amplitudes of the piston's vibration reaches a maximum the piston's motion ceases and as a result

$$\frac{d\delta}{d\tau_{\delta=\delta_{max}}} = 0.$$

From equation (3) we get

$$\sqrt{1 - (\alpha + \lambda^2\delta_{max} - \varepsilon\delta_{max})\,\mathrm{sign}\sin\tau} = 0,$$

and from this we obtain the peak to peak values of the piston's vibration

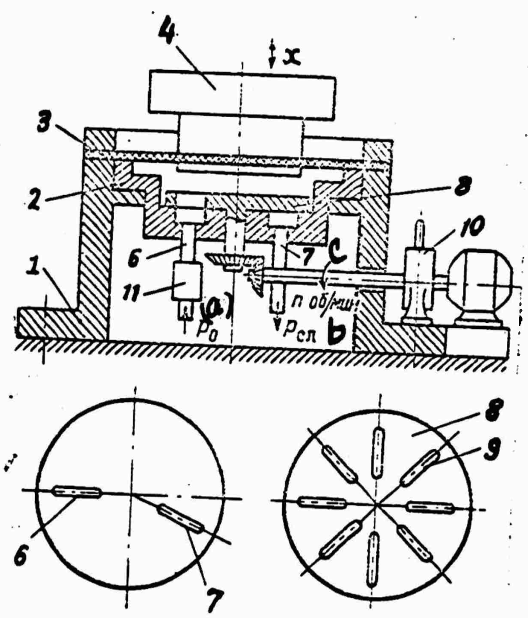
$$2\delta_{max} \approx \left(\frac{1-\alpha}{\lambda^2 - \varepsilon} + \frac{1+\alpha}{\lambda^2 + \varepsilon}\right).$$

For the example that is illustrated by figure 3 we get:

$$2\delta_{max} = \frac{1-0{,}5}{0{,}5-0{,}1} + \frac{1+0{,}5}{0{,}5+0{,}1} \approx 3{,}75 \quad 2\delta_{max} = 4).$$

In many problems of the vibration technique one has to deal with excitation of mechanical high-frequency oscillations of the given frequency and amplitude. Creation of the diverse vibration mechanisms assists in solving this question. Hydraulic and electro-hydraulic vibrators, operating on the various mineral oils, appeared recently in conjunction with a vigorous development of hydraulic drive. Basically they are pulsator-type hydraulic vibrators in which oscillations of the elastic elements, on which a load is secured, are excited by the pulsating pressure generated by a pulsating flow of liquid.

In Fig. 4, is presented the basic diagram of a vibration stand with diaphragm hydroservomotor*, in which the pulsating pressure in the working chamber of power cylinder is generated by the rotating multi-edged slide valve constituting a frequency multiplier.



a. Supply
b. Discharge
c. n/rpm

Figure 4
Basic diagram of vibration stand with diaphragm hydroservomotor.

The stand's vibrator consists of the following main parts:

- working member - diaphragm type hydraulic servo
- generator of pulsating flow - a flat revolving valve with a frequency multiplier, which is revolved by an electric motor
- frequency regulator, - hydraulic muff mounted on the supply shaft of the valve
- amplitude regulator - regulator of the supply pressure
- housing to which the power cylinder is fastened.

A table for the fastening of parts to be tested is rigidly attached to the diaphragm.

Hydraulic power cylinder (2) covered on top by an elastic membrane (3) with a platform (4) is situated on the pedestal (1).

Slit-shaped input and output openings (6) and (7) are made in the flat bottom of the cylinder, flat valve (8) with radially cut openings (9) adheres to the bottom. When the slits (9) coincide with the openings (6) oil which is fed under pressure into the power cylinder is let in and the platform is raised, when the coincide with the opening (7) the oil is let out and the platform is lowered. The valve is revolved by an electric motor through a system of gears and through the hydrodynamic mull (10). The frequency of the induced vibrations is determined by the number of slites (9) and by the number of revolutions of the valve and is regulated by means of the hydro-dynamic muff. The amplitude of the induced vibrations if determined by the supply pressure and by the mass of the loading and is regulated by valve (11).

The theory of forced vibrations of the piston of a hydraulic servo which has been developed in this article can be applied for the computation of vibrational effects in similar hydraulic vibrating mechanisms.

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FOOTNOTES

1. Page 130. N. G. Grigorovskiy, Vibratory testing stand, author's certificate No. 124179.

ON THE PROBLEM OF THE APPLICABILITY OF VOLTERRA'S DYNAMICS EQUATIONS TO NON-HOLONOMIC SYSTEMS

by

G. N. Knyazev

Holonomic couplings are imposed on the motion of a dynamic system which is described by means of 3N Cartesian coordinates $(\xi_1, \xi_2, \ldots, \xi_{3n})$ in the form

$$\frac{d\xi_\vartheta}{dt} + \sum_{s=1}^{\nu} C_{\vartheta s}\frac{d\xi_s}{dt} + \sum_{s=\nu+u+1}^{3n} C_{\vartheta\chi}\frac{d\xi_\chi}{dt} = 0 \quad (\vartheta = \nu+1, \ \nu+2, \ldots, \nu+u) \tag{1}$$

non-holonomic couplings

$$\frac{d\xi_\alpha}{dt} + \sum_{\tau=1}^{\nu+u} C_{\alpha\tau}\frac{d\xi_\tau}{dt} = 0 \qquad (\alpha = \nu+u+1 \ldots 3n), \tag{2}$$

are also imposed, where $C_{\vartheta\tau}$ and $C_{\alpha\tau}$ are functions of the system's coordinates.

It is required to write the equations of motion of such a system.

In the solution of these equations Volterra /1/ and /2/ proposes to equate to zero the following bilinear co-variants

$$\frac{d}{dt}\delta\xi_i - \delta\frac{d\xi_i}{dt} = 0 \tag{3}$$

in respect to all the Cartesian coordinates of the system $\xi_1, \xi_2, \ldots \xi_{3n}$. Expressing the derivatives of the Cartesian coordinates by means of 1 linear forms from independent non-holonomic velocities, whose number is equal to the number of degrees of freedom:

$$p_s \left(p_s \equiv \frac{d\omega_s}{dt}\right) \qquad (s=1, 2, \ldots, \nu);$$

$$\frac{d\xi_i}{dt} = \sum_{s=1}^{\nu} \xi_{is} p_s \quad (i=1, 2, \ldots, 3n), \tag{4}$$

we obtain, by using (3) the following relationships (we are using the symbols which are used in /1/:

$$\sum_{s=1}^{\nu} \xi_{is}\left(\delta p_s - \frac{d\delta\omega_s}{dt}\right) = \sum_{h=1}^{3n} \sum_{s,r=1}^{\nu} \xi_{hr} \frac{\partial \xi_{is}}{\partial \xi_h} (p_r \delta\omega_s - p_s \delta\omega_r). \tag{5}$$

Employing the Beltrami transformation

$$\delta L = \frac{d}{dt}\left(\sum_{i=1}^{3n} \frac{\partial T}{\partial \dot{\xi}_i} \delta \xi_i\right) - \delta T,$$

which follows from the D'Alembert-Lagrange equation

$$\sum_{i=1}^{3n} (X_i - m_i \ddot{\xi}_i) \delta \xi_i = 0, \tag{6}$$

where L is the Lagrange function $\xi_1, \xi_2, \ldots \xi_{3n}, \dot{\xi}_1, \ldots \dot{\xi}_{3n}$; and T is the kinetic energy of the system.

Volterra derives equations in the form

$$\frac{d}{dt}\frac{\partial T}{\partial p_s} = \sum_{r,k=1}^{\nu} a_{sk}^{(r)} \frac{\partial T}{\partial p_r} p_k + T_s + P_s \qquad (s = 1, 2, \ldots \nu), \tag{7}$$

where $a_{sk}^{(r)}$ are some functions of $\xi_1, \xi_2, \ldots, \xi_{3n}$

$$T_s = \sum_{i=1}^{3n} \frac{\partial T}{\partial \xi_i} \xi_{is};$$

P_s is a component of the generalized force in respect to $\delta\omega_s$.

In this case it is assumed that the equations are applicable both to holonomic and to non-holonomic systems. The latter was put in doubt and the discussion about the applicability of the Volterra equations to non-holonomic systems has attracted widespread attention (see /3/-/8/).

In /4/ the applicability of these equations to non-holonomic systems is denied on account of the fact that Volterra has assumed null equality of the equations (3) for all Cartesian coordinates, and in /7/ and /8/ it is deemed unapplicable on the grounds of non-coincidence of solutions in the general case /5/. However, the authors in /4/, /7/ and/8/ have not noticed the difference between the derivation of equations obtained by Volterra himself and the derivation obtained in /3/. We will carry out a derivation which was suggested by Volterra, with the difference that we will assume that (3) is satisfied only for the Cartesian coordinates $\xi_1, \xi_2 \ldots \xi_{\nu+u}$, whose number corresponds to the number of degrees of freedom of the system and of holonomic couplings.

Variations of the Cartesian coordinates have the form

$$\delta\xi_i = \sum_{s=1}^{\nu} \xi_{is}\delta\omega_s \qquad (i = 1, 2, \ldots 3n). \tag{8}$$

Variating (4) and differentiating (8) we will get the transposed relationships (3) in the form

$$\sum_{s=1}^{\nu} \xi_{rs}\left(\delta p_s - \frac{d\delta\omega_s}{dt}\right) = \sum_{h=1}^{3n} \sum_{s,k=1}^{\nu} \xi_{hk} \frac{\partial \xi_{rs}}{\partial \xi_h} (p_k \delta\omega_s - p_s \delta\omega_k) \tag{9}$$
$$(r = 1, 2, \ldots \nu + u);$$

$$\sum_{s=1}^{\nu} \xi_{as}\left(\delta p_s - \frac{d\delta\omega_s}{dt}\right) = \sum_{h=1}^{3n} \sum_{s,k=1}^{\nu} \xi_{hk} \frac{\partial \xi_{as}}{\partial \xi_h} (p_k \delta\omega_s - p_s \delta\omega_k) - \tag{10}$$
$$-\left(\delta \frac{d\xi_a}{dt} - \frac{d}{dt}\delta\xi_a\right) \qquad (\alpha = \nu + u + 1, \ldots 3n).$$

If (9) is multiplied by $m_r \xi_{rg}$ and (10) by $m_r \xi_{rg}$, as was done by Volterra, and both expressions are added we get

$$\sum_{s=1}^{\nu} E_{sg}\left(\delta p_s - \frac{d}{dt}\delta\omega_s\right) = \sum_{s,k=1}^{\nu} b_{sk}^{(g)} (p_k \delta\omega_s - p_s \delta\omega_k) - \tag{11}$$
$$- \sum_{a=\nu+u+1}^{3n} m_a \xi_{ag}\left(\delta \frac{d\xi_a}{dt} - \frac{d}{dt}\delta\xi_a\right),$$
$$(g = 1, 2 \ldots \nu)$$

where

$$E_{sg} = \sum_{l=1}^{3n} m_l \xi_{lg} \xi_{ls};$$

$$b_{sk}^{(g)} = \sum_{h=1}^{3n} \xi_{hk} \sum_{l=1}^{3n} m_l \frac{\partial \xi_{ls}}{\partial \xi_h} \xi_{lg}.$$

For the case $[E_{sg}] \neq 0$ denoting : $\frac{\partial \log \det |E_{sg}|}{\partial E_{sg}} = e_{sg}$ and solving (11) for $\delta p_s - \frac{d}{dt} \delta\omega_s$, we will obtain

$$\delta p_\gamma = \frac{d}{dt} \delta\omega_\gamma + \sum_{s,k,g=1}^{\nu} e_{g\gamma} b_{sk}^{(g)} (p_k \delta\omega_s - p_s \delta\omega_k) - \sum_{\alpha=\nu+u+1}^{3n} \sum_{g=1}^{\nu} e_{g\gamma} m_\alpha \xi_{\alpha g} \left(\delta \frac{d\xi_\alpha}{dt} - \frac{d}{dt} \delta\xi_\alpha\right). \tag{12}$$

$$(\gamma = 1, 2, \ldots \nu)$$

It is obvious that the equation (12) coincide with equalities in /1/ for the case

$$\delta \frac{d\xi_\alpha}{dt} - \frac{d}{dt} \delta\xi_\alpha = 0 \qquad (\alpha = \nu + u + 1, \ldots 3n),$$

that is for the condition that the relations (2) will be fully integrable (see for instance /8/). In this manner, the equations (7) in the form in which they were obtained by Volterra are only applicable to holonomic systems. However, if it is required to satisfy conditions

$$\sum_{\alpha=\nu+u+1}^{3n} \sum_{g=1}^{\nu} e_{g\gamma} m_\alpha \xi_{\alpha g} \left(\delta \frac{d\xi_\alpha}{dt} - \frac{d}{dt} \delta\xi_\alpha\right) = 0, \tag{13}$$

$$(\gamma = 1, 2, \ldots \nu)$$

then the equations (7) will be applicable to non-holonomic systems, which satisfy condition (13) also.

Further on Volterra, instead of (3) and of the consequent summation from 1 to 3N assumes (2): $\sum_{l=1}^{3n} m_l \xi_{lg} \left(\delta \frac{d\xi_l}{dt} - \frac{d}{dt} \delta\xi_l\right) = 0 \qquad (g = 1, 2, \ldots \nu).$

However, keeping in mind the null equality of the transposed relationships which correspond to the relations (1), these conditions represent a particular case of (13).

In this manner, the Volterra equations in the form in which they were developed by Volterra himself, are applicable to non-holonomic systems only in the case that conditions (13) are satisfied, and in the case of systems with a full number of cyclical coordinates, which are called closed non-holonomic systems in /3/, it applies only upon multiplying through and summing within the limits of the holonomic part only, that is within the limits of (9).

Nevertheless, the authors of /4/ agree that the Volterra equations are also applicable to non-holonomic systems without any additional conditions. The correct derivation of Volterra equations with marked (?) exactness is given in /3/. In contrast to Volterra the author of /3/ does not multiply relationships (5) by ξ_{ig} $(i=1,\ 2,\dots\ 3n;\ g=1,\ 2,\dots \nu)$ and does not sum them. It is postulated that the last 3n-relationships (4), in respect to the derivations of Cartesian coordinates $\frac{d\xi_\psi}{dt}(\psi=\nu+1,\dots\dots 3n$ are the equations of couplings which are imposed on the systems, in which $\frac{d\xi_\sigma}{dt}$ $(\sigma=1,\ 2,\ \dots \nu)$ are substituted for by their corresponding expressions from (4). Due to this, the relationships (4) will always be compatible (see /6/). This is the reason why the author of /3/ uses the relationships

$$\left.\begin{aligned} \frac{dx_\sigma}{dt} &= \sum_{s=1}^{\nu} E_{\sigma s} p_s, \\ \delta\frac{dx_\sigma}{dt} &= \frac{d}{dt}\delta x_\sigma \end{aligned}\right\} \qquad (\sigma=1,2,\ .\ .\ .\ \nu)$$

only relative to the independent Cartesian coordinates. The further derivation is similar to the one suggested by Volterra. The equations that were obtained are applicable to any non-holonomic systems with linear couplings. This distinction in the derivation of equations (7) which was proposed by V. V. Dobronravov and by Volterra himself, was not noticed by the authors of paper /4/. The problem of when the non-holonomic couplings should be taken into consideration, whether immedialtey after writing equations (6) or after the transformation of equations (6) into equations of the mechanics of non-holonomic systems still remains unsolved.

As an example we will present the treatment of an investigation of the stability of a gyroscopic frame with two gyroscopes, which is used for the horizontal stabilization of the pivot axis of naval anti-aircraft machine guns /10/. When the ship is rolling the stability (of the frame) is disturbed as a result of the inclination of the frame with respect to the horizontal and due to the absence of means of correction. To correct this situation there was adopted the following system, which received its input signal, proportional to the angle of rotation of the gyroscope housing relative to the inner ring, from a transducer, mounted on the inner ring of the Cardan joint. The signal was fed into a motor which turned the frame through the required angle.

A. Yu. Ishlinski in his book sets up the Lagrange equations and conducts an investigation of the stability of such a system by the Gurvits (Horowits?) method. But in the idealized following up by the tracking system, as a result of the fact that the frame does not have

any angular velocity relative to the axis ξ, there is imposed on the system a linear non-holonomic coupling, which can be written in this manner if we disregard the effects of the earth's rotation

$$\omega_{\xi_1} = \dot{\alpha} - \dot{\psi} \sin \vartheta = 0. \tag{14}$$

Due to this it is possible to use the equations of dynamics of non-holonomic systems instead of the Lagrange equations. We will introduce designation that were used in /10/:

ξ, η, ζ — coordinates system associated with the ship

(x', y', z'); (x'', y'', z'') — coordinate system associated with the frames of the gyroscopes

A-7

$\xi_1, \eta_1, \zeta_1; \xi_2, \eta_2, \zeta_2$ — coordinates systems associated with the frame

$\omega_\xi, \omega_\eta, \omega_\zeta$ — projections of the ship's velocity onto the axes associated with the ship.

The motion of the system is described by the following parameters:

α — angle of rotation of the frame in the plane $(\eta\xi)$

$\psi\vartheta$ — angles of the ship's roll and pitch

$-\beta, \beta$ — angles of inclination of the gyroscopes in the plane of the frame in the presence of coupling between the gyroscopes with a - 1 gear ratio

$-\gamma, \gamma$ — angles of rotation of the gyroscope's rotors

i — value of the current in the regulating motor

It could be seen from the schematic (fig. 1), that

$$\omega_\xi = -\dot\psi \sin\vartheta; \quad \omega_\eta = \dot\vartheta; \quad \omega_\zeta = \dot\psi\cos\vartheta. \tag{15}$$

The expression for the kinetic energy of the system which has been derived in /10/ taking (15) into consideration has the form

$$\begin{aligned} T = &\Big(\frac{1}{2} J_{\xi_1} + J_{x'}\cos^2\beta + J_{z'}\sin^2\beta + A\cos^2\beta + \\ &+ C\sin^2\beta\Big)(\dot\alpha - \dot\psi\sin\vartheta)^2 + (J_{y'} + A)\dot\beta^2 + C\dot\gamma^2 + \\ &+ 2C\dot\gamma(\dot\alpha - \dot\psi\sin\vartheta)\sin\beta + \frac{1}{2}\theta(j\dot\alpha + \dot\psi\sin\vartheta)^2 + \\ &+ \frac{1}{2}\theta'(\dot\vartheta^2 + \dot\psi^2\cos^2\vartheta) + \Big(\frac{1}{2}J_{\eta_1} + J_{y'} + A\Big)(\dot\vartheta\cos\alpha + \\ &+ \dot\psi\cos\vartheta\sin\alpha)^2 + \Big(\frac{1}{2}J_{\zeta_1} + J_{x'}\sin^2\beta + J_{z'}\cos^2\beta + \\ &+ A\sin^2\beta + C\cos^2\beta\Big)(-\dot\vartheta\sin\alpha + \dot\psi\cos\alpha\cos\vartheta)^2. \end{aligned} \tag{16}$$

Coupling (14) belongs to the type of couplings investigated by S. A. Chaplygin. Eliminating $\dot\psi$ from (16) by means of (14) we will get

$$\begin{aligned} (T) = &(J_{y'} + A)\dot\beta^2 + C\dot\gamma^2 + \frac{1}{2}\theta\dot\alpha^2(j+1)^2 + \frac{1}{2}\theta'(\dot\vartheta^2 + \\ &+ \dot\alpha^2\operatorname{ctg}^2\vartheta) + \Big(\frac{1}{2}J_{\eta_1} + J_{y'} + A\Big)(\dot\vartheta\cos\alpha + \dot\alpha\operatorname{ctg}\vartheta\sin\alpha)^2 + \\ &+ \Big(\frac{1}{2}J_{\zeta_1} + J_{x'}\sin^2\beta + J_{z'}\cos^2\beta + A\sin^2\beta + \\ &+ C\cos^2\beta\Big)(-\dot\vartheta\sin\alpha + \dot\alpha\operatorname{ctg}\vartheta\cos\alpha)^2. \end{aligned} \tag{17}$$

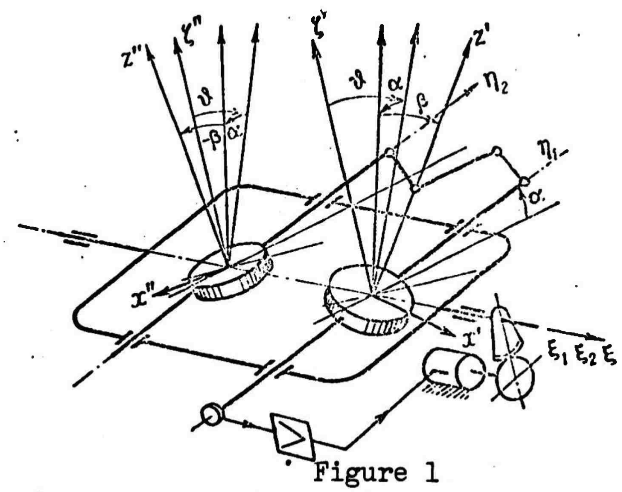

Figure 1
Schematic drawing of the gyro-frame

Having used the Dobronravov-Volterra equations we will first derive the equations of the gyro-frame by the method suggested in /3/. In Volterra designations we will assume

$$\dot{\alpha}=p_1;\ \dot{\beta}=p_2;\ \dot{\gamma}=p_3;\ \dot{\vartheta}=p_4;\ \dot{\psi}=\frac{1}{\sin\vartheta}p_1. \tag{18}$$

We will take variations

$$\delta\dot{\psi}-\frac{d}{dt}\delta\psi=0;\ \delta\dot{\beta}-\frac{d}{dt}\delta\beta=0;\quad \delta\dot{\gamma}-\frac{d}{dt}\delta\gamma=0;\quad \delta\dot{\vartheta}-\frac{d}{dt}\delta\vartheta=0, \tag{19}$$

make the derivation which was suggested by Volterra, and neglecting the moments due to friction in the gyro-grame pivot we will get the system

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial(T)}{\partial p_1}+\frac{\partial(T)}{\partial p_1}p_4\operatorname{ctg}\vartheta-\frac{\partial(T)}{\partial\alpha}&=j\frac{C_1}{g}i,\\
\frac{d}{dt}\frac{\partial(T)}{\partial p_2}-\frac{\partial(T)}{\partial\beta}&=0,\\
\frac{d}{\partial t}\frac{\partial T}{\partial p_3}-\frac{\partial(T)}{\partial\gamma}&=0,\\
\frac{d}{dt}\frac{\partial(T)}{\partial p_4}-\frac{\partial(T)}{\partial p_1}p_1\operatorname{ctg}\vartheta-\frac{\partial(T)}{\partial\vartheta}&=M_\vartheta,
\end{aligned}\right\} \tag{20}$$

to which we should add the equation of the electronic amplifier with its motor

$$-\mu\beta=Ri+L\dot{i}+jG_2\dot{\alpha}, \tag{21}$$

where j-is the gear ratio of the reducer, C_1 - coefficient of the back-EMF, g=981 cm/sec^2, M_ϑ is the moment of effect of the ship's motion on the gyro-frame which turns it through an angle ϑ ; R and L are the ohmic resistance and inductivity of the motor.

Taking into consideration (14), (16) and (17) we will set up Chaplygin's equation /11/:

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\alpha}}+\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\vartheta}-\frac{\partial(T)}{\partial\alpha}&=j\frac{C_1}{g}i,\\
\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\beta}}-\frac{\partial(T)}{\partial\beta}&=0,\\
\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\gamma}}-\frac{\partial(T)}{\partial\gamma}&=0,\\
\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\vartheta}}-\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\alpha}-\frac{\partial(T)}{\partial\vartheta}&=M_\vartheta.
\end{aligned}\right\} \tag{22}$$

The Volterra-Dobronravov equations differ from the Chaplygin equations since

$$\frac{\partial (T)}{\partial \dot{\alpha}} = \frac{\partial T}{\partial \dot{\gamma}} + \frac{\partial T}{\partial \dot{\psi}} \frac{1}{\sin \vartheta}.$$

This difference is caused by the fact that the authors introduce non-holonomic couplings in different phases of the derivation. For the problem under consideration we assume

$$\frac{\partial T}{\partial \dot{\psi}} \approx \frac{\partial (T)}{\partial \dot{\alpha}} \sin \vartheta,$$

and the Chaplygin equations (22) coincide with the Volterra-Dobronravov equations (20). Retaining only the terms of the first order in respect to the variables and their derivatives and assuming that

$$|\dot{\alpha}\dot{\vartheta}| \ll |\sin^3 \vartheta| \quad \text{и} \quad \dot{\alpha}^2 \ll |\sin^3 \vartheta|,$$

we will obtain a system in variations in the open form

$$\begin{gathered}
\ddot{\alpha}\left[\theta (j+1)^2 + J_{\eta_1} + 2J_{y_1} + 2A + \frac{1}{\sin \vartheta_0}(J_{\zeta_1} + 2J_{z'} + 2C + \theta')\right] + \\
+ \ddot{\vartheta}\left[(J_{\eta_1} + 2J_{y'} + 2A) - (J_{\zeta_1} + 2J_{z'} + 2C)\right] - 2C\dot{\gamma}\dot{\vartheta} = j\frac{C_1}{g} i; \\
2(J_{y'} + A)\ddot{\beta} = 0; \quad C\ddot{\gamma} = 0; \\
\ddot{\alpha}\left[(J_{\eta_1} + 2J_{y'} + 2A) - (J_{\zeta_1} + 2J_{z'} + 2C)\right] + \\
+ \ddot{\vartheta}(\theta' + J_{\eta_1} + 2J_{y'} + 2A) + 2C\dot{\gamma}\dot{\alpha} = 0.
\end{gathered} \tag{23}$$

From here we get

$$\beta = \beta_0 = 0; \quad 2C\dot{\gamma} = H = \text{const}.$$

Similar to the Appel equations, the equations of Volterra-Dobronravov belongs to that class of non-holonomic systems in which the couplings are taken into consideration immediatley after the writing down of the original relationship for the derivation of equations. The Chaplygin equations contain other terms, in which the non-holonomic couplings are taken into consideration only

after the equations are written in their final form and due to this fact they belong to the intermediate class of equations.

We will set up the Hamel equations, in which the couplings are taken into consideration only after the final equations are written /9/. Denoting in addition to (5)

$$p_5 = -\dot{\psi}\sin\vartheta + \dot{\alpha}, \tag{24}$$

we will get the Richchi-Hamel coefficients

$$\left.\begin{aligned} \gamma_{14}^5 &= -\operatorname{ctg}\vartheta, \\ \gamma_{41}^5 &= \operatorname{ctg}\vartheta, \\ \gamma_{45}^5 &= -\operatorname{ctg}\vartheta, \\ \gamma_{54}^5 &= \operatorname{ctg}\vartheta. \end{aligned}\right\} \tag{25}$$

The remaining coefficients are equal to zero. The Hamel equations, after taking into consideration (24) and (25) have the form

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{\alpha}} + \frac{\partial\tilde{T}}{\partial p_5}\gamma_{14}^5\dot{\vartheta} - \frac{\partial\tilde{T}}{\partial\alpha} = j\frac{C_1}{g}i; \\ &\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{\beta}} - \frac{\partial\tilde{T}}{\partial\beta} = 0; \\ &\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{\gamma}} - \frac{\partial\tilde{T}}{\partial\gamma} = 0; \\ &\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{\vartheta}} + \frac{\partial\tilde{T}}{\partial p_5}\gamma_{41}^5\dot{\alpha} - \frac{\partial\tilde{T}}{\partial\vartheta} = M_\vartheta, \end{aligned}\right\} \tag{26}$$

where $\tilde{T}=T$ upon the substitution $\dot{\alpha}$, $\dot{\vartheta}$, $\dot{\beta}$, $\dot{\psi}$ according to (18) and (24).

For these same conditions, upon opening up (26) and setting $p_5=0$ we get the equations in variations in the developed form:

$$\left.\begin{array}{c}\ddot{\alpha}\left[\Phi(j+1)^2+(J_{\eta_1}+2J_{y'}+2A)+\frac{1}{\sin^2\vartheta_0}(J_{\zeta_1}+2J_{z'}+2C+0')\right]+\\+\ddot{\vartheta}\left[(J_{\eta_1}+2J_{y'}+2A)-(J_{\zeta_1}+2J_{z'}+2C)\right]-2C\dot{\gamma}\dot{\vartheta}=j\frac{C_1}{g}i;\\2(J_{y'}+A)\ddot{\beta}=0;\\C\ddot{\gamma}=0;\\\ddot{\alpha}\left[(J_{\eta_1}+2J_{y'}+2A)-(J_{\zeta_1}+2J_{z'}+2C)\right]+\\+\ddot{\vartheta}(J_{\eta_1}+2J_{y'}+2A+0')+2C\dot{\gamma}\dot{\alpha}=0.\end{array}\right\} \quad (27)$$

It should be noted that the difference between the systems (27) and (23) does not arise even in the gyroscopic terms.

Upon integrating the three last equations of the system (23)

$$\begin{gathered}H=\text{const},\quad \beta=\beta_0=0\\\dot{\alpha}\left[(J_{\eta_1}+2J_{y'}+2A)-(J_{\zeta_1}+2J_{z'}+2C)\right]+\\+\dot{\vartheta}(J_{\eta_1}+2J_{y'}+2A+0')+H\alpha=0,\end{gathered}$$

we will get the characteristics equation of the system (23) having taken into consideration (21)

$$L(JJ''-J'^2)\lambda^4+R(JJ''-J'^2)\lambda^3+\left(j^2\frac{C_1C_2}{g}J''+H^2L\right)\lambda^2+H^2R\lambda=0,$$

where

$$\begin{gathered}J=0(j+1)^2+(J_{\eta_1}+2J_{y'}+2A)+\frac{1}{\sin^2\vartheta_0}(J_{\zeta_1}+2J_{z'}+2C+0')\\J'=(J_{\eta_1}+2J_{y'}+2A)-(J_{\zeta_1}+2J_{z'}+2C)\\J''=J_{\eta_1}+2J_{y'}+2A+0'.\end{gathered} \quad (28)$$

We will investigate the stability of the gyro-frame by the Gurvits method. As can be seen from (28), the sign of J' has no influence upon the sign of the coefficients in the characteristics equation. Coefficient $JJ''-J'^2$ is positive in the range of those moments of inertia which are used in practical applications. Accoding to the theorem proposed in /12/, the number of roots equal to zero should not exceed the number of non-holonomic couplings (in the given case one non-holonomic coupling and two roots are equal to zero) if the non-holonomic

system is to be stable. The condition of stability for the remaining roots is given by

$$j^2 C_1 C_2 R > 0,$$

that is, it is necessary to introduce a tracking system with $j \neq 0$. But as it is shown by practical experience, the tracking system is only balanced at certain gear ratios j which are given by an inequality proposed in /10/:

$$\frac{jC_2}{J_*} - \frac{\mu}{H} > 0, \tag{29}$$

where

$$J_* = J_1 + 2J_{x'} + 2A + j^2 0.$$

The statement of the given problem (ideal following by the tracking system) does not permit such an unequality. The investigation of stability using the equation

$$2(J_{y'} + A)\ddot{\beta} = 0$$

creates a critical case with a number of zero-valued roots and results in considerable difficulties in solving.

We will change the statement of the problem, assuming that there is an error in the tracking system..In this case the non-holonomic coupling takes the form

$$\dot{\alpha} - \dot{\psi}\sin\vartheta - \dot{\varepsilon} = 0. \tag{30}$$

The expression for kinetic energy (16), upon eliminating $\dot{\varphi}$ with the help of (30), will take the form

$$
\begin{aligned}
(T) = {} & (J_{y'} + A)\dot{\beta}^2 + \Big(\frac{1}{2} J_{\xi_1} + J_{x'} \cos^2\beta + J_{z'} \sin^2\beta + A\cos^2\beta + \\
& + C\sin^2\beta\Big)\dot{\varepsilon}^2 + C\dot{\gamma}^2 + 2C\dot{\gamma}\dot{\varepsilon}\sin\beta + \frac{1}{2}\theta[\dot{\alpha}(j+1) - \dot{\varepsilon}]^2 + \\
& + \frac{1}{2}\theta'[\dot{\vartheta}^2 + (\dot{\alpha} - \dot{\varepsilon})^2 \operatorname{ctg}^2\vartheta] + \Big(\frac{1}{2} J_{\eta_1} + J_{y'} + A\Big)[\dot{\vartheta}\cos\alpha + \\
& + (\dot{\alpha} - \dot{\varepsilon}) \operatorname{ctg}\vartheta \sin\alpha]^2 + \Big(\frac{1}{2} J_{\zeta_1} + J_{x'} \sin^2\beta + \\
& + J_{z'}\cos^2\beta + A\sin^2\beta + C\cos^2\beta\Big)[-\dot{\vartheta}\sin\alpha + (\dot{\alpha} - \dot{\varepsilon})\operatorname{ctg}\vartheta\cos\alpha]^2.
\end{aligned} \tag{31}
$$

We will derive the equations of the gyro-frame in the form of Volterra-Dobronravov equations. The equalities (18) and (19) will then be written respectively

$$
\dot{\alpha} = p_1;\quad \dot{\beta} = p_2;\quad \dot{\gamma} = p_3;\quad \dot{\vartheta} = p_4;\quad \dot{\varepsilon} = p_5;\quad \dot{\psi} = \frac{1}{\sin\vartheta}(p_1 - p_5) \tag{32}
$$

$$
\left.
\begin{aligned}
& \delta\dot{\beta} - \frac{d}{dt}\delta\beta = 0;\quad \delta\dot{\alpha} - \frac{d}{dt}\delta\alpha = 0;\quad \delta\dot{\gamma} - \frac{d}{dt}\delta\gamma = 0; \\
& \delta\dot{\vartheta} - \frac{d}{dt}\delta\vartheta = 0;\quad \delta\dot{\psi} - \frac{d}{dt}\delta\psi = 0.
\end{aligned}
\right\} \tag{33}
$$

Having obtained the solution suggested by Volterra, under the conditions previously mentioned we will get the system

$$
\left.
\begin{aligned}
& \frac{d}{dt}\frac{\partial(T)}{\partial p_1} - \frac{\partial(T)}{\partial p_1}\operatorname{ctg}\vartheta\, p_4 - \frac{\partial(T)}{\partial\alpha} = j\frac{C_1}{g}i, \\
& \frac{d}{dt}\frac{\partial(T)}{\partial p_2} - \frac{\partial(T)}{\partial\beta} = 0, \\
& \frac{d}{dt}\frac{\partial(T)}{\partial p_3} - \frac{\partial(T)}{\partial\gamma} = 0, \\
& \frac{d}{dt}\frac{\partial(T)}{\partial p_4} + \frac{\partial T}{\partial p_5}\operatorname{ctg}\vartheta\, p_1 - \frac{\partial(T)}{\partial p_5}\operatorname{ctg}\vartheta\, p_5 - \frac{\partial T}{\partial\vartheta} = M_\vartheta, \\
& \frac{d}{dt}\frac{\partial(T)}{\partial p_5} + \frac{\partial(T)}{\partial p_5}\operatorname{ctg}\vartheta\, p_4 - \frac{\partial(T)}{\partial\varepsilon} = M_\varepsilon.
\end{aligned}
\right\} \tag{34}
$$

Equation (21) should be added to the system (34).

From the second equation (34) we get

$$2(J_{y'}+A)\ddot{\beta}-2C\dot{\gamma}\dot{\varepsilon}=0.$$

Due to this fact, the coupling (30) belongs as before to the Chaplygin-type couplings

$$\dot{\psi}=\frac{1}{\sin\vartheta}(\dot{\alpha}-\dot{\varepsilon}). \tag{35}$$

Keeping in mind (16) and (31) we will obtain the Chaplygin-type equations

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\alpha}}+\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\vartheta}-\frac{\partial(T)}{\partial\alpha}=j\frac{C_1}{g}i;\\
&\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\beta}}-\frac{\partial(T)}{\partial\beta}=0;\\
&\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\gamma}}-\frac{\partial(T)}{\partial\gamma}=0;\\
&\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\vartheta}}-\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\alpha}+\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\varepsilon}-\frac{\partial(T)}{\partial\vartheta}=M_\vartheta;\\
&\frac{d}{dt}\frac{\partial(T)}{\partial\dot{\varepsilon}}-\frac{\partial T}{\partial\dot{\psi}}\frac{\cos\vartheta}{\sin^2\vartheta}\dot{\vartheta}-\frac{\partial(T)}{\partial\varepsilon}=M_\varepsilon.
\end{aligned} \tag{36}$$

Taking into consideration (35) we have

$$\frac{\partial(T)}{\partial\dot{\varepsilon}}=\frac{\partial T}{\partial\dot{\varepsilon}}+\frac{\partial T}{\partial\dot{\psi}}\frac{1}{\sin\vartheta}.$$

However, for the problem under consideration

$$\frac{\partial(T)}{\partial\dot{\alpha}}=-\frac{\partial T}{\partial\dot{\psi}}\frac{1}{\sin\vartheta},$$

and the equations (34) and (36) fully coincide. Keeping only the first degree terms, in respect to the variables and their derivatives, and assuming that

$$|\dot{\alpha}\dot{\vartheta}|\ll|\sin^3\vartheta_0|\,\dot{\alpha}^2\ll|\sin^3\vartheta_0|\,\dot{\vartheta}^2\ll|\sin^3\vartheta_0|$$
$$|\dot{\alpha}\dot{\varepsilon}|\ll|\sin^3\vartheta_0|,$$

we will get the system (34) in variations in the developed form

$$
\left.\begin{aligned}
&J\ddot{\alpha}+J'\ddot{\vartheta}-J_1\ddot{\varepsilon}-2C\dot{\gamma}\dot{\vartheta}=j\frac{C_1}{g}i,\\
&2(J_{y'}+A)\ddot{\beta}-2C\dot{\gamma}\dot{\varepsilon}=0,\\
&2C\ddot{\gamma}=0,\\
&J'\ddot{\alpha}+J''\ddot{\vartheta}-J'\ddot{\varepsilon}-2C\dot{\gamma}\dot{\varepsilon}+2C\dot{\gamma}\dot{\alpha}=0,\\
&-J_1\ddot{\alpha}-J'\ddot{\vartheta}+\bar{J}\ddot{\varepsilon}+2C\dot{\gamma}\dot{\vartheta}+2C\dot{\gamma}\dot{\beta}=-aj\frac{C_1}{g}\frac{di}{dt},
\end{aligned}\right\}\qquad(37)
$$

where in addition to the designations of (28) it was assumed that

$$J_1=\theta(j+1)+(J_{\eta_1}+2J_{y'}+2A)+(J_{\zeta_1}+2J_{z'}+2C+\theta')\frac{1}{\sin^2\vartheta_0};$$

$$\bar{J}=(J_{\xi_1}+2J_{x'}+2A)+(J_{\eta_1}+2J_{y'}+2A)+\theta+$$
$$+(J_{\zeta_1}+2J_{z'}+2C+\theta')\frac{1}{\sin^2\vartheta_0};$$

where a is the proportionally coefficient.

As was shown by the proof, the Hamel equations fully coincide with (37) in this case also. Upon integration of the expressions of the system (37) and upon adding the correction equation (21) and upon assuming that $J_1=$ and writing the characteristics equation of the system

$$
\begin{aligned}
&HL(JJ''-J'^2)\lambda^4+(JJ''-J'^2)\left(HR-aj\frac{C_1}{g}\mu\right)\lambda^3+\\
&+\left[H^3L+j\frac{C_1}{g}\mu(\bar{J}J''-J'^2)+j^2\frac{C_1C_2}{g}HJ''\right]\lambda^2+\\
&+\left(-aj\frac{C_1}{g}\mu+HR\right)H^2\lambda+j\frac{C_1}{g}\mu H^2=0,
\end{aligned}\qquad(38)
$$

we will get, according to Gurvits, the following stability conditions:

$$R>\frac{\mu a}{H}j\frac{C_1}{g};\qquad\frac{jC_2}{J^*}>\frac{\mu}{H};\qquad J^*=\bar{J}-J,\qquad(39)$$

which coincide with the stability conditions which were obtained in /10/, with the exception of the sign of the term J*.

In contrast to /10/ we have obtained a condition imposed upon the ohmic resistance of the motor.

It can be seen from (39) that (jC_2/J^*) is maximum at

$$j = \sqrt{\frac{J_{\xi_1} + 2J_{x'} + 2A}{0}}.$$

In this manner, we have clarified the difference in the manner in which the equations are derived by Volterra and the manner in which this was done by V. V. Vobronravov, and we have proven the correctness of the derivations of the equations which has been presented in /3/. The derivation of the equations of motion for the gyro-frame, treating it as a non-holonomic system, has shown that the Chaplygin and Volterra-Dobronravov equation fully coincide, for the case of a solution with an error in the tracking system that was assumed in this article. The investigation of the stability of a gyro-frame that has been presented in this article hss resulted in imposing a more strict condition (39) than the condition of stability that has been proposed in /10/. In the course of solving the problem we have shown the full coincidence between the equations of these types and the Hamel type equations. The question of timeliness of using non-holonomic couplings has been left unresolved, as before.

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INVESTIGATION OF THE STABILITY OF STEADY-STATE MOTION OF A SERVO WITH FEEDBACK, WHICH IS BEING ACTED UPON BY A LOAD, TAKING INTO CONSIDERATION THE COMPRESSIBILITY OF THE OIL

by

G. N. Knyazev

This paper persents an investigation, by the Lyapunov method, of the stability of steady-state motion of a servo with feedback, that is being acted upon by a load, taking into consideration the compressibility of the oil and the deformation of the piplines. A schematic drawing of a typical servo system under load is shown in fig. 1.

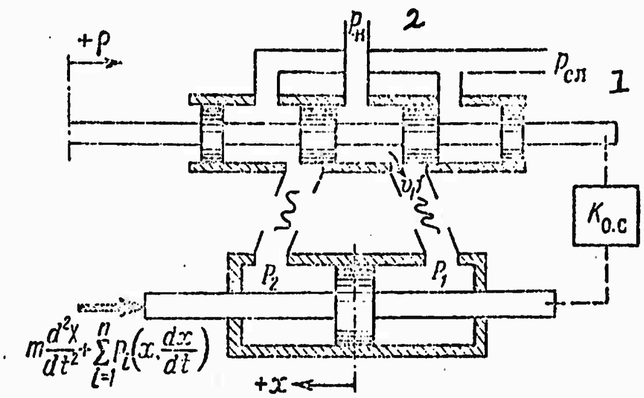

Figure 1
Schematic drawing of a hydraulic motor with feedback; 1) discharge (subscript); 2) supply (subscript).

The following assumptions are made: the fluid is supplied to the slide valve at constant pressure, there is nol loss of fluid in the motor and in the valve, the coefficient of oil flow through the valve opening is constant, the pressure drop in the chambers of the servo does not exceed the value of pressure in the pressure supply pipeline the temperature of the fluid entering the valve is constant.

Initial Equation of Motion of the Hydraulic Motor

We will first derive the equations of motion of a hydraulic motor without feedback. When the compressibility of the oil and the deformation of pipelines are taken into consideration, the increase in the capacity is determined by the difference in the quantities of fluid that enter the cavity which connects the hydraulic motor with the slide valve, and upon the quantity of fluid is displaced from this cavity in unit time.

The continuity equations taking as positive the direction shown in fig. 1, have the form

$$\left.\begin{aligned} \frac{dV_1}{dt} &= v_1 f_1 - F\frac{dx}{dt}, \\ \frac{dV_2}{dt} &= F\frac{dx}{dt} - v_2 f_2, \end{aligned}\right\} \quad (1)$$

where V_1; V_2 – are the volumes of the right and left cavity of the hydraulic motor, respectively.

A-1

F – is the area of the cylinder

$\frac{dx}{dt}$ – velocity of the piston rod displacement

$f_1 f_2$ – are the respective areas of the flow-conducting apertures of the slide valve

$v_1 v_2$ – are the average velocities of fluid flow in the respective flow-conducting areas.

As was shown in /1/, for the case of symmetrically constructed slide valve

we have

$$f_1 = f_2 = f;\ v_1 = v_2 = v.$$

Subtracting the second equation (1) from the first, we get

$$\frac{dV_1}{dt} - \frac{dV_2}{dt} + 2F\frac{dx}{dt} = 2vf. \tag{2}$$

Assuming that the supply pipes are elastic and identically the same, we have

$$\frac{dV_1}{dt} = k\frac{dP_1}{dt}; \qquad \frac{dV_2}{dt} = k\frac{dp_2}{dt}, \tag{3}$$

where p_1 and p_2 are the pressures in the respective cavities of the motor, and k is the coefficient of eaasticity of the pipe system and of the fluid is present in the cavities of the system.

If the motion of the motor with no load ($p_1=p_2$) is considered, then the equation (1) taking into consideration /1/ has the form

$$F\frac{dx}{dt} = \mu\sqrt{\frac{g}{\gamma}}\, f\sqrt{p_s - p_d} \tag{4}$$

where p_s is the pressure in the supply main, p_d is the discharge pressure, and μ is the (oil) consumption coefficient.

When the valve aperture is rectangular, and if we take into consideration the changes in the pressure and discharge areas of the motor which occur as a result of the valve displacement to either side from its neutral position, the equations (4) will take the form given in /2/

$$F\frac{dx}{dt} = \mu l\sqrt{\frac{g}{\gamma}}\sqrt{|(p_s - p_d)\,\mathrm{sign}\,\rho|}\ \rho, \tag{5}$$

where l is the length of the valve aperture and ρ is the valve displacement.

The pressure drop in the chambers of the motor changes under the influence of internal forces in the manner given by

$$\Delta p = p_1 - p_2 = \frac{m\frac{d^2x}{dt^2} + \sum_{i=1}^{n} P_i\left(x, \frac{dx}{dt}\right)}{F}, \tag{6}$$

where m is the mass that has been brought to the piston rod of the cylinder and $\sum_{i=1}^{n} P_i\left(x, \frac{dx}{dt}\right)$ is the summation of forces, which depend upon the displacement and velocity of the hydraulic cylinder (summation of forces is assumed to be equal to the mathematical expectancy of a statistical load, and as a it is assumed to be a time differentiable function).

Under the influence of the load upon the motor equation (2), on taking into consideration (3), (5) and (6) will take the form

$$k\frac{d\Delta p}{dt} + 2F\frac{dx}{dt} = \\ = 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{|(p_н - p_{сл})\,\mathrm{sign}\,\rho - \Delta p|}\,\rho. \tag{7}$$

In our further investigation it will be more convenient to rewrite equation (7) in the form

$$k\frac{d\Delta p}{dt} + 2F\frac{dx}{dt} = \\ = 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{|(p_н - p_{сл}) - \Delta p\,\mathrm{sign}\,\rho|}\,\rho. \tag{8}$$

The motion of a hydraulic servo with feedback is defined by the equation of the open-circuited working mechanism

and by the feedback equation

$$k\frac{d\Delta p}{dt} + 2F\frac{dx}{dt} = 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{|p_0 - \Delta p\,\mathrm{sign}\,s|}\,s \tag{9}$$

$$\varepsilon = p - k_{o.c}x, \tag{10}$$

where $p_o = p_s - p_d$, ε is the tracking error and $k_{o.c.}$ is the feedback coefficient.

Since in the case when the hydraulic servo acts as a motor we have the condition

$$p_0 > \Delta p \operatorname{sign} \varepsilon,$$

we get in the right side of (9) an analytical function Δp for the case when $\varepsilon \gtrless 0$, which function can be expanded in a Taylor series in the region of non-turbulent flow at Δp^* and ε^*. If we limit ourselves to terms of the first order, we will get

$$\begin{aligned} k\frac{d\Delta p}{dt} + 2F\frac{dx}{dt} &= 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{p_0 - \Delta p^* \operatorname{sign}\varepsilon^*}\,\varepsilon^* + \\ &+ 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{p_0 - \Delta p^* \operatorname{sign}\varepsilon^*}\,(\varepsilon - \varepsilon^*) - \\ &- \mu l\sqrt{\frac{g}{\gamma}}\frac{\varepsilon^* \operatorname{sign}\varepsilon^*}{\sqrt{p_0 - \Delta p^* \operatorname{sign}\varepsilon^*}}(\Delta p - \Delta p^*). \end{aligned} \tag{11}$$

For laminar flow this equation takes the form

$$k\frac{d\Delta p^*}{dt} + 2F\frac{dx^*}{dt} = 2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{p_0 - \Delta p^* \operatorname{sign}\varepsilon^*}\,\varepsilon^*. \tag{12}$$

Subtracting (12) from (11) and introducing the designations

$$x - x^* = \eta;$$
$$\varepsilon - \varepsilon^* = \sigma,$$

we will get, upon taking (6) into consideration the variational equation of motion of the hydraulic servo,

$$\frac{km}{F}\frac{d^3\eta}{dt^3}+\mu l\sqrt{\frac{g}{\gamma}}\frac{|\varepsilon^*|}{\sqrt{p_0-\Delta p^*\operatorname{sign}\varepsilon^*}}\frac{m}{F}\frac{d^2\eta}{dt^2}+2F\frac{d\eta}{dt}=$$
$$=2\mu l\sqrt{\frac{g}{\gamma}}\sqrt{p_0-\Delta p^*\operatorname{sign}\varepsilon^*}\sigma-$$
$$-\frac{\mu l}{F}\sqrt{\frac{g}{\gamma}}\frac{|\varepsilon^*|}{\sqrt{p_0-\Delta p^*\operatorname{sign}\varepsilon^*}}\sum_{l=1}^{n}\left[P_l\left(x,\frac{dx}{dt}\right)-P_l\left(x^*,\frac{dx^*}{dt}\right)\right]-$$
$$-\frac{k}{F}\sum_{l=1}^{n}\frac{d}{dt}\left[P_l\left(x,\frac{dx}{dt}\right)-P_l\left(x^*,\frac{dx^*}{dt}\right)\right]. \tag{13}$$

The feedback equation for laminar flow has the form

$$\varepsilon^*=\rho^*-k_{o.c}x^*. \tag{14}$$

The feedback equation in variation form will be obtained by subtracting (14) from (10) and assuming that the function ρ can not be variated

$$\sigma=-k_{o.c}\eta.$$

Taking into consideration that all $P_l\left(x,\frac{dx}{dt}\right)$, are non-decreasing functions of the coordinates and velocity of the servo, we obtain on the left side of equation (13) the function $f(\sigma, t)$ which belongs to the class (A) at $0<t$ (when a gap exists in the system, or to the class (A') /3/). For any $P_l\left(x,\frac{dx}{dt}\right)$ which acts upon the servo, it is necessary that the summation of the functions which is expressed by the equation

$$f_1(\sigma,t)=-\left\{\mu l\sqrt{\frac{g}{\gamma}}\frac{|\varepsilon^*|}{\sqrt{p_0-\Delta p^*\operatorname{sign}\varepsilon^*}}\sum_{l=1}^{n}\left[p_l\left(x,\frac{dx}{dt}\right)-\right.\right.$$
$$\left.\left.-p_l\left(x^*,\frac{dx^*}{dt}\right)\right]+k\sum_{l=1}^{n}\frac{d}{dt}\left[p_l\left(x,\frac{dx}{dt}\right)-p_l\left(x^*,\frac{dx^*}{dt}\right)\right]\right\}.$$

should belong to the functions of the given classes.

Finally we will get

$$\left.\begin{aligned}&\frac{d^3\eta}{dt^3}+\frac{\mu l}{k}\sqrt{\frac{g}{\gamma}}\frac{|\varepsilon^*|}{\sqrt{p_0-\Delta p^*\operatorname{sign}\varepsilon^*}}\frac{d^2\eta}{dt^2}+\frac{2F^2}{km}\frac{d\eta}{dt}=f(\sigma,t),\\&\sigma=-k_{o.c}\eta.\end{aligned}\right\} \tag{15}$$

We will denote

$$\frac{\mu l}{k}\sqrt{\frac{g}{\gamma}}\,\frac{|\varepsilon^*|}{\sqrt{p_0-\Delta p^*\,\mathrm{sign}\,\varepsilon^*}}=a;\quad \frac{2F^2}{km}=c.$$

we will make our investigation with a constant. Particular cases of this kind of motion will be motions with ε^*=const, Δp^*=const with slow and weak changing of the coefficients ε^* and Δp. These operating conditions prevail in heavy machine building, during the rolling and pitching of a ship and in the tracking of an object.

We will write the equation (15) in the form

$$\frac{d^3\eta}{dt^3}+a\frac{d^2\eta}{dt^2}+c\,\frac{d\eta}{dt}=f(\sigma)$$
$$\sigma=-k_{o.c}\eta \qquad (16)$$

The investigation of stability of the linearized equation (16) for $f(\sigma)=h\sigma$, by the Gurvits method shows, that stability is preserved only at a limited load:

$$h<\frac{2aF^2}{kk_{o.c}m}. \qquad (17)$$

As was shown in /4/, the trivial solution $\eta=0$, will be assimptotically stable for any initial disturbances if

$$a>0;\ f(0)=0,\ \frac{f(\eta)}{\eta}>0 \text{ при } \eta\neq 0,$$

$$\lim_{r\to\infty}\omega\left(\eta,\frac{d\eta}{dt}\right)=\infty, \text{ где } r=\sqrt{\eta^2+\left(\frac{d\eta}{dt}\right)^2};$$

$$\omega\left(\eta,\frac{d\eta}{dt}\right)=a\int_0^{\eta}f(\eta)\,d\eta+f(\eta)\frac{d\eta}{dt}+\frac{F^2}{km}\left(\frac{d\eta}{dt}\right)^2;$$

or

$$\frac{df(\eta)}{d\eta}<\frac{2aF^2}{km} \qquad (18)$$

$$\frac{df(\sigma)}{d\sigma} < \frac{2\mu l F^2}{k^2 k_{o.c} m} \sqrt{\frac{g}{\gamma}} \frac{|s^*|}{\sqrt{p_0 - \Delta p^* \operatorname{sign} s^*}}. \tag{18}$$

Equation (18) shows the coincidence of the linearized equation with the condition of stability (17).

For condition (18) it is possible to investigate the influence of the parameters and modes of operation of the tracking system on its stability for a given load or to see how and for which change in loading pattern do the regions of stability of the system change. It is interesting to note that as the compressibility of the oil increases, the upper limit of load change decreases as k^2. For the case of incompressible oil ($k=0$) the system is stable for any load change.

In order to increase the possibility of application of the tracking system for the case of changing loads, which exceed the limit given by (18), we will introduce feedback as a function of the integral and the first derivative of the signal

$$\sigma = \rho - k_{o.c}\left(x + k_1 \int_{t_0}^{t} x\,dt + T\frac{dx}{dt}\right)$$

or in variations

$$\sigma = -k_{o.c}\left(\eta + k_1 \int_{t_0}^{t} \eta\,dt + T\frac{d\eta}{dt}\right),$$

where T is a time constant of the differentiating loop, and k_1 is the transfer coefficient of the integrating loop.

We will write the equation (16) in the Lur'ye form

(See page 160a) (19)

$$
\left.
\begin{aligned}
\frac{d\eta}{dt} &= \eta_1, \\
\frac{d\eta_1}{dt} &= \eta_2, \\
\frac{d\eta_2}{dt} &= -a\eta_2 - \frac{2F^2}{km}\eta_1 + f(\sigma), \\
\sigma &= -k_{o.c}\left(\eta + k_1\int_{t_0}^{t_1}\eta\,dt + T\eta_1\right).
\end{aligned}
\right\}
$$

The roots of the system are

$$\lambda_1=0;\ \lambda_{2,3}=-\frac{a}{2}\pm\sqrt{b};\ b=\frac{a^2}{4}-\frac{2F^2}{km},$$

from where it follows that the system is inherently stable in two coordinates and neutral in one coordinate.

We will put (19) into canonical form by a linear transformation:

$$\begin{aligned} x_1&=C_1^1\eta+C_2^1\eta_1+C_3^1\eta_2;\\ x_2&=C_1^2\eta+C_2^2\eta_1+C_3^2\eta_2;\\ x_3&=C_1^3\eta+C_2^3\eta_1+C_3^3\eta_2, \end{aligned}$$

in which the coefficients $C_k^{(s)}$ we will determine by the formula

$$C_k^{(s)}=\frac{1}{H_i(\lambda_s)}D_{ik}(\lambda_s),\quad (k,s=1,2,3)\ [\text{см. (2.23) в [3]}]$$

where

$$H_i(\lambda_s)=\sum_{k=1}^{3}h_kD_{ik}(\lambda_s)\quad (s=1,2,3);$$

h_k are the coefficients of $f(\sigma)$ in (19); $D_{ik}(\lambda_s)$ are the algerbraic co-factors of the i-th column of the determinant of the system (19). We have

$$h_1=0;\ h_2=0;\ h_3=1;$$

$$D_{11}(\lambda)=\lambda^2+a\lambda+\frac{2F^2}{km};$$

$$D_{12}(\lambda)=\lambda+a;$$

$$D_{13}(\lambda)=1;$$

$$C_1^1=\frac{2F^2}{km};\quad C_2^1=a;\quad C_3^1=1;$$

$$C_1^2=0;\quad C_2^2=\frac{a}{2}+\sqrt{b};\quad C_3^2=1;$$

$$C_1^3=0,\quad C_2^3=\frac{a}{2}-\sqrt{b};\quad C_3^3=1.$$

Solving for η by Cramer's method we have

$$\left.\begin{aligned}\eta &= \frac{km}{2F^2}x_1 - \frac{km}{4F^2\sqrt{b}}\left(\frac{a}{2}+\sqrt{b}\right)x_2 + \frac{km}{4F^2\sqrt{b}}\left(\frac{a}{2}-\sqrt{b}\right)x_3,\\ \eta_1 &= \frac{1}{2\sqrt{b}}x_2 - \frac{1}{2\sqrt{b}}x_3,\\ \eta_2 &= -\frac{\left(\frac{a}{2}-\sqrt{b}\right)}{2\sqrt{b}}x_2 + \frac{\left(\frac{a}{2}+\sqrt{b}\right)}{2\sqrt{b}}x_3.\end{aligned}\right\} \qquad (20)$$

The system of equations in the canonical form will be written as follows

(See page 162a)

(21)

The derivative of σ in respect to t in canonical transofrmation will have the form

$$\begin{aligned}\frac{d\sigma}{dt} &= -k_{o.c}k_1\frac{km}{2F^2}x_1 - \\ &- k_{o.c}\left[\frac{1}{2\sqrt{b}} - \frac{kk_1m}{4F^2\sqrt{b}}\left(\frac{a}{2}+\sqrt{b}\right) - T\frac{\left(\frac{a}{2}-\sqrt{b}\right)}{2\sqrt{b}}\right]x_2 - \\ &- k_{o.c}\left[-\frac{1}{2\sqrt{b}} + \frac{kk_1m}{4F^2\sqrt{b}}\left(\frac{a}{2}-\sqrt{b}\right) + T\frac{\left(\frac{a}{2}+\sqrt{b}\right)}{2\sqrt{b}}\right]x_3.\end{aligned} \qquad (22)$$

We take a standard absolutely positive function (see /3/):

$$V = \frac{1}{2}Ax_1^2 - \frac{a_2^2}{2\lambda_2}x_2^2 - \frac{a_3^2}{2\lambda_3}x_3^2 - \frac{2a_2a_3}{\lambda_2+\lambda_3}x_2x_3 + \int_0^\sigma f(\sigma)\,d\sigma, \qquad (23)$$

where $A > 0$, a_2 and a_3 are real if $b > 0$ and are complex conjugates if $b < 0$.

In accordance with the equations (21) and (22) we find the derivative of V in respect to time. In order for dV/dt to be a negative function it is sufficient that

$$A - k_{o.c}k_1\frac{km}{2F^2} = 0;$$

$$-\frac{a_2^2}{\lambda_2} - \frac{2a_2a_3}{\lambda_2+\lambda_3} - \frac{k_{o.c}}{2\sqrt{b}} + \frac{k_{o.c}kk_1m}{4F^2\sqrt{b}}\left(\frac{a}{2}+\sqrt{b}\right) + k_{o.c}T\frac{\left(\frac{a}{2}-\sqrt{b}\right)}{2\sqrt{b}} = 0; \qquad (24)$$

$$-\frac{a_3^2}{\lambda_3} - \frac{2a_2a_3}{\lambda_2+\lambda_3} + \frac{k_{o.c}}{2\sqrt{b}} - \frac{k_{o.c}kk_1m}{4F^2\sqrt{b}}\left(\frac{a}{2}-\sqrt{b}\right) - k_{o.c}T\frac{\left(\frac{a}{2}+\sqrt{b}\right)}{2\sqrt{b}} = 0. \qquad (25)$$

$$\left.\begin{aligned}\frac{dx_1}{dt}&=f(\sigma),\\ \frac{dx_2}{dt}&=\lambda_2 x_2+f(\sigma),\\ \frac{dx_3}{dt}&=\lambda_3 x_3+f(\sigma).\end{aligned}\right\}$$

Multiplying (24) by λ_2 and (25) by λ_3 and adding we get

$$-(a_2+a_3)^2-k_{o.c}+k_{o.c}Ta=0,$$

from where

$$T>\frac{1}{a}=\frac{k}{\mu l}\sqrt{\frac{\gamma}{g}}\frac{\sqrt{p_0-\Delta p^*\operatorname{sign}\dot{s}^*}}{|\dot{s}^*|}. \tag{26}$$

Multiplying (24) by $\frac{1}{\lambda_2}$, (25) by $\frac{1}{\lambda_3}$ and adding, we will get

$$-\left(\frac{a_2}{\lambda_2}+\frac{a_3}{\lambda_3}\right)^2+\frac{k_{o.c}km}{2F^2}-\frac{k_{o.c}k^2k_1m^2a}{4F^4}=0,$$

from where

$$0<k_1<\frac{2F^2}{amk}. \tag{27}$$

In actuality the stability will not be disturbed if we will make $k_1=0$. In this manner it is possible to introduce into the systems of machine tools only the first derivative of the input signal.

Conclusions

1. When the influence of compressibility of the oil and of the deformations of the piplines is taken into consideration, the equation of a loaded hydraulic working mechanism with feedback contains a derivative of the third order, as a result of this it is possible to preserve the stability during the steady-state operation only with a limited range of velocity of load variation (see (18)).

2. In order to stabilize a tracking system in steady-state or near to it when the servo is acted upon by any velocity-variable loads

which belong to the class A or A' it is necessary to introduce a differentiating circuit with a time constant which is given by formula (26). It is recommended to introduce an additional feedback element which will regulate as a function of an integral with the coefficient given by foruula (27). It should be kept in mind that the stability conditions given by (26) and (27) are only necessary (but not necessarily sufficient trans).

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CLASSIFICATION OF THE MOTION OF A HEAVY RIGID BODY CONTAINING A STATIONARY POINT

By

V. A. Chikin

We will examine a rigid body S which is fastened at the stationary point O. As a stationary reference system we will use a Cartesian system of coordinates $O\xi\eta\zeta$ with the axis $O\zeta$ directed vertically up. As a moving reference system which is rigidly tied to the rigid body, we will choose the main inertial axes Oxyz.

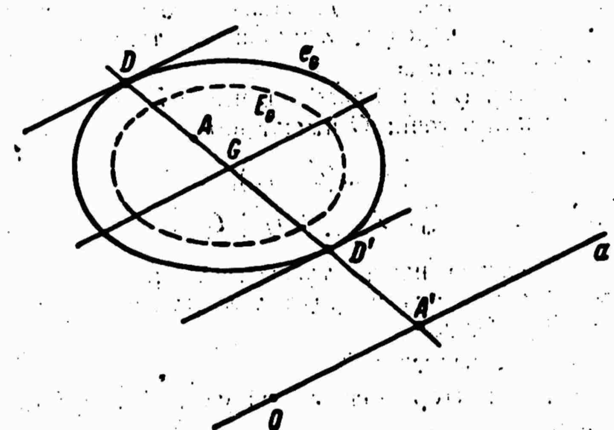

Figure 1

Let E_G be a section of the central ellipsoid of inertia cut in the plane in which the center gravity of the body is lying and in which there also is some straight line a which passes through the point O (fig. 1). Following the method used by Signorini /1/, /2/, we will examine the ellipse e_G similar to the ellipse E_G, with a center at point G and with the similarity coefficient $\frac{\pi}{\sigma_E\sqrt{m}}$, where σ_E is the area of E_G and m is the mass of the body. This ellipse is

customarily called the central ellipse of inertia of the straight line a. Further, through the center of gravity G in the plane of the central ellipse we will draw a diameter DD' which is crossed by a diameter parallel to the straight line a. The point A, lying on DD' and displaced from G in direction opposite to the point of intersection of DD' with the straight line a, by a distance

$$GA = \frac{GD^2}{GA'}, \tag{1}$$

was called by Signorini the anti-pole of the straight line a relative to the central ellipse. He has also proven a theorem stating that for a body moving around the stationary point O, the system of sliding vectors corresponding to the values of motion (distances) of each point of the body reduce in the general case to a screw (spiral?) whose axis passes through the anti-pole A_1 of the instantaneous axis of rotation of the central ellipse of inertia, perpendicular to the plane of the sllipse.

Using the Signorini theorem, we will find the coordinates of A_1 in the major axes of inertia, in the form of coordinates of the point of intersection of the axis of the spiral by the system of vectors representing the quantities of motion of the body's points with the plane Π_ω, that passes through the center of gravity of the body G and the vector of the instantaneous velocity ω.

The main vector $\bar{Q}$ and the main moment $\bar{K}_0$ of the system of vectors of the quantities of motion at the point O have the form

$$\bar{Q} = m(\bar{\omega} \times \bar{r}_0), \tag{2}$$

$$\bar{K}_0 = Ap\bar{i} + Bq\bar{j} + Cr\bar{k}, \tag{3}$$

where $\bar{r}_G$ of the radius vector of the center of gravity of the body

A, B, C are the moments of inertia in respect to the major ases (A-1) of the Oxyz coordinates system,

$\bar{i}, \bar{j}, \bar{K}$ are units vectors

p, q, r are the projections of the vector of instantaneous angular velocity $\bar{\omega}$.

We will reduce the given system to a spiral. The equation of the spiral axis can be written in the form

$$\bar{K}_0 - \bar{r} \times \bar{Q} = \lambda \bar{Q}, \tag{4}$$

where $\bar{r}$ is the radius vector of the point situated on the spiral axis.

Multiplying, starting from the left, the left and right sides of equation (4) by $\bar{Q}$ we will get

$$\bar{Q} \times \bar{K}_0 - \bar{r}Q^2 + \bar{Q}(\bar{r}\bar{Q}) = 0. \tag{5}$$

The vector $\bar{Q}$ is perpendicular to the plane $\Pi\omega$. We will denote the radius vector of the point of intersection of the axis with this plane (that is of the anti-pole A_1) by $\bar{r}_1$; since $\bar{r}_1$ is perpendicular to $\bar{Q}$, we will get from equation (5)

$$\bar{r}_1 = \frac{\bar{Q} \times \bar{K}_0}{Q^2} \tag{6}$$

Projecting (6) onto the major inertial axes, it is possible to obtain the coordinates to the anti-plle of the instantaneous axis of revolution relative to the central ellipse. The projection of the anti=pole A_1 on the direction of the instantaneous axis of rotation is equal to

$$\bar{r}_1 \bar{\omega}^\circ = \frac{(\bar{Q} \times \bar{K}_0)\bar{\omega}^\circ}{Q^2}, \tag{7}$$

where $\bar{\omega}^{\circ}$ is the unit vector of the instantaneous axis of rotation.

In the further discussion we will, for the sake of brevity, call the anti-pole of the instantaneous axis of rotation relative to the central ellipse the first anti-pole and we will denote it by the letter A_1. In addition, we will introduce for consideration the anti-pole of an axis that coincides with the vector of instantaneous angular acceleration relative to the central ellipse we will call this anti-pole the second anti-pole and we will denote it by the symbol A_2. This anti-pole is defined similarly to the first and is situated in the plane π_{ε} that passes through the vector of instantaneous angular acceleration and through the center of gravity of the body.

Examining further the problem of reducing the centrifugal and rotational inertial forces to the simplest form, we will establish a number of interesting properties of certain modes of motion of a rigid body around a stationary point, which properties we will formulate in the following theorems.

Theorem 1. When a rigid body possessing a stationary point is in motion, then at any given instant of time the system of vectors of the centrifugal inertial forces reduces to a dynamic spiral whose axis passes through the first anti-pole in a direction perpendicular to the instantaneous axis of rotation; the system of vectors of turning inertial forces also reduces to a dynamic spiral whose axis passes through the second anti-pole perpendicualr to the plane π_{ε}, in which this anti-pole is situated.

As is well known, the major vector and major moment of a system of centrifugal inertial forces are equal, respectively, to

$$\overline{R}^{n} = -m\overline{W}_{0}^{n} = -m(\overline{\omega}\times\overline{v}_{0}) = -\overline{\omega}\times\overline{Q}, \tag{8}$$

$$\overline{M}_{0}^{n} = \overline{\omega}\times\overline{K}_{0}. \tag{9}$$

The equation of the axis of the dynamic spiral can be written in the form

$$\overline{M}_{0}^{n} - \overline{r}\times\overline{R}^{n} = \lambda_{1}\overline{R}^{n}. \tag{10}$$

It can be seen from (8) and (9) that this axis is perpendicular to the axis of the vector of $\overline{\omega}$. We will prove that the axis of the dynamic spiral passes through the first anti-pole, A_1. In order to do this we will represent the radius vector of a point on the axis as a sum of two vectors

$$\overline{r} = \overline{r}_{B} + \lambda_{2}\overline{n}, \tag{11}$$

where $\overline{r}_B$ is the radius vector of the point of intersection of the dynamic axis of the spiral with the instantaneous axis of rotation, and $\overline{n}$ is a unit-vector of the direction to be found.

Now the equation (10) can be written in the form

$$\overline{M}_{0}^{n} - \overline{r}_{B}\times\overline{R}^{n} = \lambda_{1}\overline{R}^{n}. \tag{12}$$

Multiplying vectorially, starting from the left, both sides of this equality by $\overline{R}''$ and substituting the values of $\overline{R}^n$ and $\overline{M}_o^n$ according to equations (8) and (9), we will find

$$\overline{r}_{B} = \frac{\overline{\omega}[\overline{K}_{0}(\overline{\omega}\times\overline{Q})]}{\omega^{2}Q^{2}}. \tag{13}$$

Upon projecting (13) on the direction of $\overline{\omega}$, we will get

$$\overline{r}_{B}\overline{\omega}^{\circ} = \frac{(\overline{Q}\times\overline{K}_{0})\overline{\omega}^{\circ}}{Q^{2}} = \overline{r}_{1}\overline{\omega}^{\circ}. \tag{14}$$

This proves the first part of the theorem.

We shall now examine the system of vectors of the inertial rotational forces. The major vector and major moment of this system are, as is well known, expressed by the formulas

$$\bar{R}^{\tau} = -m\bar{W}_0^{\tau} = -m(\dot{\bar{\omega}} \times \bar{r}_0) = -\dot{\bar{Q}}'; \tag{15}$$

$$\bar{M}_0^{\tau} = -(A\dot{p}\bar{i} + B\dot{q}\bar{j} + C\dot{r}\bar{k}) = -\dot{\bar{K}}_0. \tag{16}$$

where $\dot{\bar{Q}}$ and $\dot{\bar{K}}_0$ are the respective derivatives of the vectors of the quantities of motion and the kinetic moment of the body.

The major vector $\bar{R}^{\tau}$ is, according to theorem (15), perpendicular to the plane Π_{ε}, which passes through the vector of angular acceleration $\dot{\omega}$ and through the center of gravity of the body. Formulas (15) and (16) can be obtained from formulas (2) and (3) if derivatives in respect to time are substituted instead of p,q and r. It follows that the location of the second anti-pole, in its role as the point of intersection of the dynamic axis of turning forces with the plane Π_{ε} will be given by a formula similar to (6)

$$\bar{r}_2 = \frac{\dot{Q}' \times \dot{\bar{K}}_0}{(\dot{Q}')^2}, \tag{17}$$

which proves, the theorem.

Theorem 2. In order that for a case of rigid body rotating around a stationary point the systems of vectors of rotational and centrifugal inertial forces should be respectively reducible to their resultant forces at any given moment of time, it is necessary and sufficient that the vector of the

instantaneous angular velocity and of instantaneous angular acceleration should belong to the cone of permanent axes of rotation.

As is well known, the equation of the cone of permanent axes of rotation on the major inertial axes has the form

$$(B-C)x_G yz+(C-A)y_G zx+(A-B)z_G xy=0. \tag{18}$$

The necessary and sufficient condition for reducing a system of vectors to a resultant vector is the null-equality of the scalar product of the major vector by the major moment of the given system. We will write this condition for the rotational and centrifugal inertial forces.

For rotational inertial forces

$$\overline{R}^{r}\overline{M}_0^{r}=0,$$

On the basis of (15) and (16) we find

$$(\dot{\overline{\omega}}\times\overline{r}_O)\overline{K}_0=(B-C)x_O\dot{q}\dot{r}+(C-A)y_O\dot{r}\dot{p}+(A-B)z_O\dot{p}\dot{q}=0. \tag{19}$$

For centrifugal inertial forces

$$\overline{R}^{c}\overline{M}_0^{c}=0.$$

From (8) and (9) and the Lagrange indentity, taking into consideration that $\overline{Q}$ is perpendicualr to $\overline{\omega}$, we find

$$(\overline{\omega}\times\overline{Q})(\overline{\omega}\times\overline{K}_0)=\omega^2(\overline{Q}\overline{K}_0)-(\overline{Q}\overline{\omega})(\overline{K}_0\overline{\omega})=\omega^2(\overline{Q}\overline{K}_0). \tag{20}$$

from where

$$(B-C)x_Gqr+(C-A)y_Grp+(A-B)z_Gpq=0. \tag{21}$$

Equations (19) and (21) demonstrate the fact that the vectors of the instantaneous angular acceleration $\dot{\bar{\omega}}$ and of the instantaneous angular velocity $\bar{\omega}$ belong to the cone given by equation (18). In this manner, the theorem is proven.

Theorem 3. In the case of a rigid body that is rotating around a stationary point the system of vectors of the centrifugal inertial forces reduces to resultant couple or to a dynamic helix, if an only if, the system of vectors of quantities of motion of the points of the body is reducible to a resultant couple or a helix.

Let the system of vectors $m_\nu\bar{v}_\nu$ of the particles comprising the body be reducible to a couple, that is

$$\bar{Q}=0; \quad \bar{K}_0\neq 0. \tag{22}$$

In order to reduce the system of vectors of the centrifugal forces to a couple it is necessary that

$$\bar{R}^{s}=-\bar{\omega}\times\bar{Q}=0; \quad \bar{M}_0^{s}\neq 0, \tag{23}$$

and since $\bar{\omega}$ is perpendicular to $\bar{Q}$, it follows that the product $\bar{Q}\times\bar{\omega}$ can only be equal to zero if $\bar{Q}=0$ which coincides with condition (22).

The condition for reducing the system of vectors $m_\nu\bar{v}_\nu$ to a resultant vector has the form

$$\bar{Q}\bar{K}_0=(B-C)x_Gqr+(C-A)y_Grp+(A-B)z_Gpq=0, \tag{24}$$

but according to equation (20) this condition is at the same time also a condition for the reduction of the system of vectors of centrifugal inertial forces to a resultant force.

If we exclude from consideration the examined cases, we convince ourselves of the truth of the statement of the theorem about the reduction of the system of vectors under examination to a helix.

Theorem 4. If in the case of a rigid body rotating around a stationary point, the system of vectors of the quantities of motions of the particles comprising the body is reducible to a resultant force, then the fourth algebraic integral of the equations of motion of an integral of the form

$$\bar{K}_0\bar{r}_G = Apx_G + Bqy_G + Crz_G = \text{const.} \tag{25}$$

We will write the dynamic equations of motion of a heavy rigid body around a stationary point, in vectorial form

$$\dot{\bar{K}}_0 + \bar{\omega}\times\bar{K}_0 = mg(\bar{\zeta}^\circ\times\bar{r}_G), \tag{26}$$

where $\bar{\zeta}^\circ$ is a unit vector of vertical line pointing upward, and g is the acceleration of gravity.

Multiplying the left and right parts of this equation scalarly by $\bar{r}_G$ we get

$$\dot{\bar{K}}_0\bar{r}_G + \bar{r}_G(\bar{\omega}\times\bar{K}_0) = 0. \tag{27}$$

According to (24)

$$\bar{r}_0(\bar{\omega}\times\bar{K}_0)=\bar{K}_0(\bar{r}_0\times\bar{\omega})=-\frac{1}{m}\bar{K}_0\bar{Q}=0$$

and it follows that the relationship

$$\dot{\bar{K}}_0\bar{r}_0=\frac{d}{dt}(Apx_0+Bqy_0+Crz_0)=0$$

gives the fourth integral of the equation (26)

$$Apx_0+Bgy_0+Crz_0=\text{const.}$$

In this manner, in all cases of motion of a heavy rigid body around a stationary point which satisfy the conditions of theorem 4, the fourth algebraic integral is linear and characterizes the constancy of the projection of the kinetic moment vector on the direction of the line that connects the stationary point and the center of gravity of the body. The instantaneous axis of rotation, according to theorem 2 belongs in these cases to the cone of permanent axes.

On the basis of established properties, it is proposed to classify all the known modes of motion of a heavy rigid body around a stationary point into three classes, depending upon the most simple form to which the system of vectors of quantities of motion is reducible; to a resultant force, couple or helix.

In accord with this classification the modes of motion that satisfy the condition

$$(B-C)x_Gqr+(C-A)y_Grp+(A-B)z_Gpq=0. \qquad (24)$$

belong to the first class when the system of vectors $m_v\bar{v}_v$ is reducible to a resultant force.

An examination of this relationship shows that the dynamically permissible case of motion, which belong to the first class will be the following:

1. The Lagrange case (1836)

$$A=B;\ x_G=y_G=0. \tag{28}$$

2. The case of full dynamic symmetry

$$A=B=C. \tag{29}$$

3. Hesse's case (1890)

$$A(B-C)x_G^2-C(A-B)z_G^2=0;\ \ y_G=0. \tag{30}$$

4. Permanent rotations of Mlodziyevskiy-Shtaude (1894)

$$(B-C)x_Gqr+(C-A)y_Grp+(A-B)z_Gpq=0\ \ (p,\ q,\ r=\text{const}) \tag{31}$$

5. The V. A. Steklov-D. Bobylev case (1893)

$$B=2A;\ \ x_G=z_G=0;\ \ r=0;\ \ q=q_0. \tag{32}$$

The equations of motion for all these cases have a fourth algebraic integral of the form (21). Investigation of the condition (24) support the results obtained by Chaplygin in /4/, where he shows that a linear integral of the form (21) can not exist under circumstances other than those presented.

To the second class of modes of motion, in the case when the system of vectors $m_v\bar{v}_v$ reduce to a couple, should belong the modes of motion which satisfy the equality

$$\bar{Q}=m(\bar{\omega}\times\bar{r}_G)=0. \tag{33}$$

An investigation of this condition shows that the dynamically permissible modes of motion of this class will be only those for which

$$x_G = y_G = z_G = 0, \tag{34}$$

that is we come to the Euler case (1750).

To the third class we will put the rest of known cases, in which the system of vectors $m_v \bar{v}_v$ reduces to a helix. To this class belong, in part, the following:

1. The S. V. Kovalevskaya case (1888)

$$A = B = 2C; \; y_G = z_G = 0. \tag{35}$$

2. The second V. A. Steklov case (1899)

$$B > A > 2C; \; y_G = z_G = 0. \tag{36}$$

3. The D. A. Goryachev-S. A. Chaplygin case (1899)

$$A = B = 4C; \; y_G = z_G = 0. \tag{37}$$

4. The second D. A. Goryachev case (1899)

$$AC = 8(A - 2B)(B - C); \; y_G = z_G = 0. \tag{38}$$

5. The second S. A. Chaplygin case (1904)

$$0{,}6 > \frac{C}{A} > 0{,}5965; \quad 1{,}5 < \frac{B}{A} < 1{,}5965; \quad y_G = z_G = 0. \tag{39}$$

6. Kovalevskiy case (1907)

$$AC = 9(A - 2B)(B - C), \; y_G = z_G = 0 \tag{40}$$

7. Grioli case (1947)

$$(B - C)\,x_G^2 - (A - B)\,z_G^2 = 0, \quad y_G = 0; \quad \omega = \text{const}. \tag{41}$$

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GEOMETRICAL PROPERTIES OF ANTI-POLE FOR THE CASE OF A HEAVY, NON-SYMMETRICAL BODY MOVING AROUND A STATIONARY POINT IN THE GRIOLI CASE

By

V. A. Chikin

Let Oxyz be a system of mutually perpendicualr coordinates axes with unit-vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$, and let it be rigidly coupled to the body at the stationary point O in such a manner that the axis Oy coincides with mean semi-axis of the ellipsoid of inertia and the axis Oz coincides with the perpendicular to one of the circular cross-sections of the given ellipsoid. Then the equation of the ellipsoid will have the form

$$A(x^2+y^2)+Cz^2-2Ezx=1, \qquad (1)$$

where A and C are the axial (moments of inertia trans) and E is the centrifugal moment of inertia.

If the center of gravity of the body lies on the axis Oz, that is, on the perpendicular to the circular cross-section (1), then the Euler-Poisson equations have a solution. This solution on the major inertial axes was obtained by M. P. Gulyayev /3/. The dynamic possibility of existence of such types of motion was first proven in 1947, by Grioli /1/ and the motions proper acquired the name of regular precession of a heavy non-symmetrical body.

We shall take the solution of the Euler-Poisson equations a they are written in reference /4/ (formulas (40) and (41)). Not subtracting from the generality of the discussions we will introduce the constant $\tau_1=\frac{\pi}{2}$ whereupon it will be possible to write these solutions in the form

$$
\left.\begin{aligned}
p &= n\sin nt & \gamma_1 &= \frac{n^2}{Mgz_G}(C\sin nt + E\cos^2 nt),\\
q &= n\cos nt & \gamma_2 &= \frac{n^2}{Mgz_G}(C - E\sin nt)\cos nt,\\
r &= n & \gamma_3 &= \frac{n^2}{Mgz_G}E\sin nt,
\end{aligned}\right\} \qquad (2)
$$

where according to the above quoted reference /4/

$$n=\sqrt[4]{\frac{(Mgz_G)^2}{C^2+E^2}} \qquad (3)$$

is the angular velocity of precession, p,q, and r are the projections of angular velocity $\bar{\omega}$ of the body on the axes chosen, z_G is the coordinate of the center of gravity of the body, M is the mass of the body and g is the acceleration of gravity.

It follows from (2) that the moving axois is represented by the cone

$$x^2+y^2-z^2=0 \qquad (4)$$

with its vertex situated at the roigin of the coordinates. The axis of this cone coincides with the axis Oz, that is, with the perpendicular to circular cross-section (1), and the generatrix are inclined by an angle β $=45^\circ$ to the axis Oz.

We will find the projections of the quantity of motion Q and the kinetic moment $\bar{K}_O$ of the body relative to a stationary point on the axis Oxyz.

$$
\left.\begin{aligned}
Q_x &= Mnz_G\cos nt; & K_x &= n(A\sin nt - E);\\
Q_y &= -Mnz_G\sin nt; & K_y &= nA\cos nt\\
Q_z &= 0; & K_z &= n(C - E\sin nt).
\end{aligned}\right\} \qquad (5)
$$

Since

$$\bar{Q}\bar{K}_0 = Mn^2 z_0 E \cos nt \neq 0, \tag{6}$$

we come to the conclusion that the system of vectors of the quantity of motion of the points of the body reduces to a helix.

Following the results of the preceeding article, the point of intersection of the central axis of this helix with the plane α_ω which plane passes through the center of gravity of the body with the vector of instantaneous velocity ω is the anti-pole of rotation relative to the central ellipse of inertia in the plane α_ω. In our further discussion we will, for the sake of brevity call this point the first anti-pole A_1 and we will denote its radius-vector by $\bar{r}_1$. Keeping in mind that the vector $\bar{Q}$ is perpendicular to the plane α_ω we will find the location of the first anti-pole of the body, using for this purpose the equation of the central axis of the system of vectors

$$\bar{r}_1 = \frac{\bar{Q} \times \bar{K}_0}{Q^2}. \tag{7}$$

Substituting the values of quantities according to (5) we will get

$$\bar{r}_1 = \frac{1}{Mz_0}[(E \sin nt - C)(\bar{i} \sin nt + \bar{j} \cos nt) + (A - E \sin nt)\bar{k}]. \tag{8}$$

From (8) we get the coordinates of the kinetic center in the Oxyz coordinate system

$$\left.\begin{aligned} x_1 &= \frac{1}{Mz_0}(E \sin nt - C) \sin nt, \\ y_1 &= \frac{1}{Mz_0}(E \sin nt - C) \cos nt, \\ z_1 &= \frac{1}{Mz_0}(A - E \sin nt). \end{aligned}\right\} \tag{9}$$

Eliminating the time t from (9) we will get

$$x_1^2+y_1^2-\left[z_1-\frac{1}{Mz_0}(A-C)\right]^2=0 \qquad (10)$$

which is an equation of a circular cone with a vertex N, situated on the axis Oz at a distance

$$ON=\frac{1}{Mz_0}(A-C). \qquad (11)$$

from the point O.

The generatrix of this cone make an angle $\beta=45^\circ$ with its axis which coincides with the axis Oz.

We will now relate the coordinates system OXZ to the plane $\alpha\omega$ in such a manner that the axis OZ should coincide with the axis OZ, and the axis OX will be oriented along the line of intersection of the plane $\alpha\omega$ with the plane Oxy. In this case the position of the first anti-pole in the plane $\alpha\omega$ is defined by the vector

$$\bar{r}_1=\frac{1}{Mz_0}[(E\sin nt-C)\bar{c}^\circ+(A-E\sin nt)\bar{k}], \qquad (12)$$

where $\bar{c}^\circ$ is a unit vector of the axis OX.

In this manner

$$\left.\begin{aligned} X_1&=\frac{1}{Mz_0}(E\sin nt-C)\\ Z_1&=\frac{1}{Mz_0}(A-E\sin nt)\end{aligned}\right\} \qquad (13)$$

and

represent the coordinates of the kinetic center in the OXZ coordinate system.

Eliminating time from (13) we will get the equation of the generatrix of the cone (10)

$$X_1 + Z_1 = \frac{1}{Mz_G}(A - C), \tag{14}$$

which is in accord with the results of reference /2/.

It follows that in the Grioli case the first anti-pole of the body moves along the straight line given by equation (14) which rotates at uniform velocity in the body, describing a circular cone during the time $T_1 = \frac{2\pi}{n}$. Since the motion along the generatrix of equation (14) has the same period T_1, then the curve which is described by the first anti-pole on the cone (10) is a closed one. This curve is contained between two parallels which appear on the cone as a result of the fact that the cone is bisected by the planes

$$Z_1 = \frac{1}{Mz_G}(A - E) \quad \text{и} \quad Z_1 = \frac{1}{Mz_G}(A + E), \tag{15}$$

The distance from the stationary point O to the vertex N of the cone (10) depends upon the axial moments of inertia A and C and the distance z_G as can be seen from equation (11). Applying the Huygens-Shteiner theorems, this distance can be expressed by means of moments of inertia relative to the central axis which are parallel to the axes Oxyz. If we will denote the moment of inertia relative to the axis that passes through G parallel to Ox and A*, then (11) will take the form

$$ON = \frac{1}{Mz_G}(A^* - C + Mz_G^2), \tag{16}$$

from where

$$ON - OG = \frac{1}{Mz_G}(A^* - C). \tag{17}$$

It follows from (17) that the ponts N and O will lie on different sides of G for a body for which $A^* < C$, for any position of the pivot point on the axis Oz. If $A^* < C$, they the point N will lie on the same side of G as the point O. Finally, in the case when $A^*=C$, the point N coincides with the center of gravity of the body for any position of the point O on the axis Oz. This case corresponds to the case of a body in which the planes of circular sections of the central ellipsoid are mutually orthogonal.

Let us now assume that the body is pivoted, instead of at point O, at the point N which does not coincide with the center of gravity G and it happens to be the vertex of the cone (10). In this case it follows from (16) and (17) that the vertex of the cone that contains the first anti-pole will coincide with the point O and its generatrix will coincide with the generatrix of the former axoid (4) and the cone (10) will become a stationary axoid. In this manner we can formulate the following theorem:

In the case when a heavy rigid body moves around the point O and for this body the circular sections of the central ellipsoid are not mutually orthogonal and the body satisfies the Grioli conditions, the stationary axoid (4) and its respective cone (10) which contains at any given instant the first anti-pole, are reciprocal.

In the case of orthogonality of the circular sections of the central ellipsoid of inertia there is no sense in talking about the reciprocity of the cones (4) and (10) since, in accordance with (3), $z_G=0$, $n=0$ and consequently $\omega=0$.

In the preceeding article (see page 138 of the original) we have introduced the concept of the anti-pole of an axis which coincides with the vector of instantaneous angular acceleration $\bar{\varepsilon}$ relative to the central ellipse in the plane α_ε, which plane passes through the center of gravity of the body and through the vector of angular acceleration. For the sake of brevity we will further call this anti-pole the second anti-pole of the body. The second anti-pole is the point of intersection of the central axis of the system of vectors of the turning inertial forces with the plane α_ε.

In order to find this position in the body, let us set up the major vector $\bar{R}^\tau$ and the major moment $\bar{M}_0^\tau$ of the turning inertial forces at the stationary point. As is well known, they are respectively equal to the respective time derivatives of the vectors of the quantity of motion and of the kinetic moment, taken with an opposite sign.

$$\bar{R}^\tau = -\dot{\bar{Q}}^\tau; \quad \bar{M}_0^\tau = -\dot{\bar{K}}_0. \tag{18}$$

In this manner, using (5) we get

$$\left.\begin{aligned} R_x^\tau &= Mn^2 z_G \sin nt; & M_x^\tau &= -n^2 A \cos nt; \\ R_y^\tau &= Mn^2 z_G \cos nt; & M_y^\tau &= n^2 A \sin nt; \\ R_z^\tau &= 0; & M_z^\tau &= n^2 E \cos nt. \end{aligned}\right\} \tag{19}$$

From (19) it follows directly that

$$R^\tau = Mn^2 z_G \quad \text{и} \quad \bar{R}^\tau \bar{M}_0^\tau = 0,$$

that is for a motion of a heavy rigid body in the Grioli case the turning inertial force are reducible to the resultant force $\bar{R}^\tau$ applied at the second anti-pole of the

body perpendicular to the plane $\alpha\varepsilon$.

When the turning forces are reduced to a single resultant force, the angular acceleration vector should belong to the cone of permanent axes. This cone disintegrates in the Grioli case into two planes, one of which is the major plane, and is perpendicular to the median semi-axis of the inertial ellipsoid; and the second is the plane of circular section of the ellipsoid which is now under discussion. From what has been said above and from the formulas (2) it follows that for the motion of a body in this case the vector of instantaneous angular acceleration $\bar{\varepsilon}$ remains at any instant of time in the plane of circular section of the inertial ellipsoid, which section is perpendicular to the line connecting the center of gravity of the body with the stationary point.

The position of the second anti-pole we will find from a formula similar to (7)

$$\bar{r}_2 = \frac{\dot{Q}' \times \bar{K}_0'}{(\dot{Q}')^2} = \frac{1}{Mz_G}\left[E\cos nt\,(\bar{i}\cos nt - \bar{j}\sin nt) + A\bar{k}\right], \tag{20}$$

consequently, its coordinates will be

$$\left.\begin{aligned} x_2 &= \frac{E}{2Mz_G}(1-\cos 2nt), \\ y_2 &= \frac{-E}{2Mz_G}\sin 2nt, \\ z_2 &= \frac{A}{Mz_G}. \end{aligned}\right\} \tag{21}$$

Eliminating time from (21), we will get

$$\left.\begin{aligned} \left(x_2 - \frac{E}{2Mz_G}\right)^2 + y_2^2 &= \left(\frac{E}{2Mz_G}\right)^2, \\ z_2 = \frac{A}{Mz_G} &= \frac{\rho_x^2}{z_G}, \end{aligned}\right\} \tag{22}$$

that is, in the case of motion of the body in the Grioli case, the second antipole is displaced in the body around a periphery situated on a circular cylinder with a radius $\frac{E}{2Mz_G}$ having an axis passing parallel to the axis Oz through a point having coordinates $\left(\frac{E}{2Mz_G}, 0, 0\right)$. The plane of this circle is removed from the plane of circular section of the inertial ellipsoid by the distance $\frac{\rho_x^2}{z_G}$ where ρ_x is the radius of inertia of the body relative to an axis lying in the plane of this circular section.

We will examine a particular case of motion of a body that satisfies the Grioli conditions. Let the structure of the body be such that

$$A^* > C. \tag{23}$$

From G we will draw a perpendicular to one of the circular sections of the central ellipsoid, and the the distance

$$GO = z_G = \sqrt{\frac{A^* - C}{M}} \tag{24}$$

we will fasten the body at the point O. Then, according to (16)

$$ON = 2z_G, \tag{25}$$

that is, the vertex N of the cone and the point O will be situated at different sides of G and at the same distance defined by the equality (24). The moments of inertia of the body in respect to the axis Ox and an axis passing through N parallel to Ox, will be equal. In this case we come to the conclusion, that not only the cones (4) and (10) will be mutually reciprocal, but the regular precessions performed by the body if it is fastened at the point O or at the point N are also reciprocal.

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GENERAL EQUATIONS OF MOTION OF A RIGID BODY WITH LIQUID-FILLED CAVITIES, AND THE GENERALIZATION OF ONE THEOREM DUE TO N. E. ZHUKOVSKIY

by

I. S. Chikina

Basic Propositions of Tensor Calculus

The derivation of equations of motion of a complex mechanical system, such system being a rigid body having liquid filled cavities is made much easier if use is made of the methods of vector and tensor calculus. This is due, not only to the fact that the utilization of vector and tensor symbolisms makes it easier to present voluminous analytical dissertations, but also to the fact that it permits us to uncover various new physical facts and to deduct the respective relationships and functions faster and more implicitly.

Besides the generally known basic propositions from the field of algebra and from the field of analysis of affine orthogonal tensors, we will introduce auxiliary operations necessary for the obtaining of some results in the following article.

We will examine the affine orthogonal vector of the second rank in the three dimensional space; components of the tensor will be denoted by α_{ij}, the entire tensor will be denoted by α and we will write it in the form of a matrix

$$\alpha \equiv \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix} \equiv \alpha_{ij} \qquad (i, \quad j=1, 2, 3). \tag{1}$$

As usual we will call the tensor

$$\alpha+\beta\equiv\alpha_{ij}+\beta_{ij}. \tag{2}$$

the sum of the tensors α and β .

The scalar product $\alpha\beta=\gamma$ of the two vectors α and β with the components α_{ij} and β_{lk} we will call a tensor whose components will be obtained from the components of the given tensors in the same manner as we obtain the elements of a matrix expressing the product of two matrices, that is, by multiplying the rows of the first matrix by the columns of the other.

In other words, the elements of the tensor γ will be expressed by means of the elements of α and β such that:

$$\gamma_{kl}=\sum_{r=1}^{3}\alpha_{kr}\beta_{rk} \qquad (k,\ l=1,\ 2,\ 3). \tag{3}$$

From the above follows the obvious concept of the entire positive power α^n of the given matrix.

We will further examine the vector

$$\bar{a}=a_x\bar{i}+a_y\bar{j}+a_z\bar{k}. \tag{4}$$

We will call the scalar multiplication of the tensor α by the vector $\bar{a}$ the operation of "action" of the tensor α on the given vector $\bar{a}$, which results in a new vector which we will denote by $\alpha\bar{a}$. This new vector has the following form

$$\alpha\bar{a}=(\alpha_{11}a_x+\alpha_{12}a_y+\alpha_{13}a_z)\bar{i}+(\alpha_{21}a_x+\alpha_{22}a_y+\alpha_{23}a_z)\bar{j}+ \\ +(\alpha_{31}a_x+\alpha_{32}a_y+\alpha_{33}a_z)\bar{k} \tag{5}$$

and will, generally speaking, be non-colinear with the vector $\bar{a}$. In

this manner, the tensor α is defined by giving the three values which it assumes for the three given non-complanar vectors. Thus, if the given vectors $\bar{i}, \bar{j}, \bar{k}$ are known, and $\alpha\bar{i}, \alpha\bar{j}, \alpha\bar{k}$ are known, then the tensor is fully defined. In parts, if in accordance to the definition of "action", we should act by the tensor α on the three non-complanar vectors $\bar{i}, \bar{j}, \bar{k}$ then we will get three new vectors

(See page 191a) (6)

Each of the new vectors $\bar{A}$, $\bar{B}$ and $\bar{C}$ is in the general case non-colinear with any of the given vectors.

We respectively associate with the three non-complanar vectors $\bar{i}(1, 0, 0)$, $\bar{j}(0, 1, 0)$, $\bar{k}(0, 0, 1)$ three vectors

$$\bar{A}(\alpha_{11}, \alpha_{21}, \alpha_{31});\quad \bar{B}(\alpha_{12}, \alpha_{22}, \alpha_{32}),\quad \bar{C}(\alpha_{13}, \alpha_{23}, \alpha_{33}),$$

whose components define the tensor α.

We will call the matrix

$$K = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{7}$$

the unit matrix. We will introduce the operation of "multiplication" of matrices K and α with the understanding that this really means according to (3) the obtaining of a matrix transposed in respect to the matrix α so that:

$$K\alpha \quad \begin{Vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{Vmatrix}, \tag{8}$$

The following, easily proven relationships follow from the previously made definitions:

$$\left.\begin{aligned}\alpha\bar{i} &= \alpha_{11}\bar{i} + \alpha_{21}\bar{j} + \alpha_{31}\bar{k} = \bar{A};\\ \alpha\bar{j} &= \alpha_{12}\bar{i} + \alpha_{22}\bar{j} + \alpha_{32}\bar{k} = \bar{B};\\ \alpha\bar{k} &= \alpha_{13}\bar{i} + \alpha_{23}\bar{j} + \alpha_{33}\bar{k} = \bar{C}.\end{aligned}\right\}$$

$$KK\alpha = \alpha; \tag{9}$$

K^2 is a unit matrix

(10)

(11)

$$K^n = K;$$
$$K\alpha^n = (K\alpha)^n. \tag{12}$$

We will introduce certain concepts which are related to the field of vector and tnesor analysis.

By the derivative $d\alpha/dx$ of the given matrix α in respect to the scalar argument we will understand a matrix whose elements are derivatives in respect to the x-argument of the respective elements of the given matrix α considering the matrix to be a function of the point (x, y, z):

$$\frac{\partial \alpha}{\partial x} \equiv \frac{\partial \alpha_{ij}}{\partial x}; \qquad (i, j = 1, 2, 3). \tag{13}$$

We will establish the concept of a gradient of the given matrix α, understanding it to be a vector set up according to the rule:

$$\operatorname{grad} \alpha = \frac{\partial \alpha}{\partial x}\bar{i} + \frac{\partial \alpha}{\partial y}\bar{j} + \frac{\partial \alpha}{\partial z}\bar{k}. \tag{14}$$

where by $\frac{\partial \alpha}{\partial x}$, $\frac{\partial \alpha}{\partial y}$, $\frac{\partial \alpha}{\partial z}$ are understood the above mentioned martices whose elements are the derivatives in respect to x, y and z of the respective elements of the matrix α .

By the divergence of vector $\bar{b}$ we will understand the usual expression

$$\operatorname{div} \bar{b} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}. \tag{15}$$

If $\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}$ is a vector with components independent of x, y and z, then it is possible to prove the following relationship:

$$\operatorname{grad} \alpha \cdot \bar{a} = \operatorname{div} \{(K\alpha) \cdot \bar{a}\}. \qquad (16)$$

It is possible to establish the Laplace operation, applicable to the tensor

$$\Delta\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2}, \qquad (17)$$

where by $\frac{\partial^2\alpha}{\partial x^2}, \frac{\partial^2\alpha}{\partial y^2}, \frac{\partial^2\alpha}{\partial z^2}$ are understood tensors whose elements are the second derivatives in respect to the scalar arguments x, y and z of the respective elements of the matrix α.

If we should take a vector $\bar{a}$ with components which are functions of x, y and z, that is

$$a_x = a_x(x, y, z),$$
$$a_y = a_y(x, y, z),\ a_z = a_z(x, y, z),$$

then in respect to this vector we will establish another operation Δ' a, understanding it to be a vecotr, set up according to the following rule:

$$\Delta'\bar{a} = \Delta'(a_x\bar{i} + a_y\bar{j} + a_z\bar{k}) = \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2}\right)\bar{i} + \left(\frac{\partial^2 a_y}{\partial x^2} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial z^2}\right)\bar{j} + \left(\frac{\partial^2 a_z}{\partial x^2} + \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_z}{\partial z^2}\right)\bar{k}. \qquad (18)$$

In other words, the vector $\Delta'\bar{a}$ can be called the "laplacian" of the vector $\bar{a}$.

We will stop to consider certain operations which transform a matrix into a scalar, and in part we will derive the known expressions of the invariants of the tensor.

We will denote the operation of setting up a sum of scalar products of the vectors $\bar{i}, \bar{j}, \bar{k}$ by the vectors $\alpha\bar{i}, \alpha\bar{j}, \alpha\bar{k}$ by the symbol $I_1\alpha$ that is

$$I_1\alpha = \bar{i}\cdot\alpha\bar{i} + \bar{j}\cdot\alpha\bar{j} + \bar{k}\cdot\alpha\bar{k}.$$

Upon performing the multiplication, we will get

$$\bar{i}\cdot\alpha\bar{i} = \bar{i}\cdot(\alpha_{11}\bar{i} + \alpha_{21}\bar{j} + \alpha_{31}\bar{k}) = \alpha_{11}$$

and similarly,

$$\bar{j}\cdot\alpha\bar{j} = \alpha_{22} \quad \bar{k}\cdot\alpha\bar{k} = \alpha_{33},$$

and consequently

$$I_1\alpha = \alpha_{11} + \alpha_{22} + \alpha_{33}, \tag{19}$$

where $\alpha_{11}, \alpha_{22}, \alpha_{33}$ — are the diagonal elements of the matrix α.

We will examine the vector-scalar product if the vectors:

$$(\alpha\bar{j}\times\alpha\bar{k})\cdot\bar{i}; \quad (\alpha\bar{k}\times\alpha\bar{i})\cdot\bar{j}; \quad (\alpha\bar{i}\times\alpha\bar{j})\cdot\bar{k},$$

where by the symbol $\times$ is meant vectorial product.

We have

$$\alpha\bar{j}\times\alpha\bar{k} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} = \bar{i}\begin{vmatrix} \alpha_{22} & \alpha_{32} \\ \alpha_{23} & \alpha_{33} \end{vmatrix} + \bar{j}\begin{vmatrix} \alpha_{32} & \alpha_{12} \\ \alpha_{33} & \alpha_{13} \end{vmatrix} + \bar{k}\begin{vmatrix} \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \end{vmatrix},$$

from where

$$(\alpha\bar{j}\times\alpha\bar{k})\cdot\bar{i} = \begin{vmatrix} \alpha_{22} & \alpha_{32} \\ \alpha_{23} & \alpha_{33} \end{vmatrix}.$$

Similarly we will obtain

$$(\alpha\bar{k}\times\alpha\bar{i})\cdot\bar{j} = \begin{vmatrix} \alpha_{33} & \alpha_{13} \\ \alpha_{31} & \alpha_{11} \end{vmatrix}; \quad (\alpha\bar{i}\times\alpha\bar{j})\cdot\bar{k} = \begin{vmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{vmatrix}.$$

Each of the determinants of the second order can be called a "dyad". We will denote the sum of three dyads by $I_2\alpha$, that is

$$I_2\alpha = (\alpha\bar{j} \times \alpha\bar{k})\cdot\bar{i} + (\alpha\bar{k} \times \alpha\bar{i})\cdot\bar{j} + (\alpha\bar{i} \times \alpha\bar{j})\cdot\bar{k}$$

or

$$I_2\alpha = \begin{vmatrix} \alpha_{22} & \alpha_{32} \\ \alpha_{23} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{13} \\ \alpha_{31} & \alpha_{11} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{vmatrix}. \tag{20}$$

We will examine the mixed product of the vectors $\alpha\bar{i} \times \alpha\bar{j})\alpha\bar{k}$, which is a determinant, composed of the elements of the tensor α.

If we will denote this operation by $I_3\alpha$, then

$$I_3\alpha = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}. \tag{21}$$

We will define in additional detail the derivative of a vector which is applied at any point of a space along the radius-vector of that point.

Let, at the point M, be applied a vector $\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}$, (a_x, a_y, a_z) are functions of x, y and z and let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ be the radius-vector of the point M. The vector $\bar{a}$ is a function of the radius-vector of the point. Let us find the derivative of $\bar{a}$ in respect to $\bar{r}$. In order to do this we will increase the radius vector by $\Delta\bar{r}$, that is we will move over to the point M', whose radius-vector is $\bar{r}'$, as a result

$$\bar{r}' - \bar{r} = \Delta\bar{r} = \Delta x\bar{i} + \Delta y\bar{j} + \Delta z\bar{k}.$$

Let there be a vector $\bar{a}'$ applied to the point M', that is the vector $\bar{a}$ will receive an increment $\Delta\bar{a} = \bar{a}' - \bar{a}$.

Expanding the increment of each of the components in a Taylor series and noticing that

$$\Delta\bar{a} = \Delta a_x\bar{i} + \Delta a_y\bar{j} + \Delta a_z\bar{k}$$
$$\Delta x = \Delta\bar{r}\cdot\bar{i};\quad \Delta y = \Delta\bar{r}\cdot\bar{j};\quad \Delta z = \Delta\bar{r}\cdot\bar{k},$$

we will get

$$\Delta\bar{a} = \frac{\partial\bar{a}}{\partial x}(\Delta\bar{r}\cdot\bar{i}) + \frac{\partial\bar{a}}{\partial y}(\Delta\bar{r}\cdot\bar{j}) + \frac{\partial\bar{a}}{\partial z}(\Delta\bar{r}\cdot\bar{k}) + \bar{\varepsilon}.$$

We will divide both parts of this expression by $\Delta\ \bar{r}$, multiply it by an arbitrary vector $\bar{v}$ and we will take the limit at $\Delta\bar{r}\to 0\,(M'\to M)$:

$$\lim_{\Delta\bar{r}\to 0}\left(\frac{\Delta\bar{a}}{\Delta\bar{r}}\bar{v}\right) = \frac{\partial\bar{a}}{\partial x}(\bar{v}\cdot\bar{i}) + \frac{\partial\bar{a}}{\partial y}(\bar{v}\cdot\bar{j}) + \frac{\partial\bar{a}}{\partial z}(\bar{v}\cdot\bar{k}) + \lim_{\Delta\bar{r}\to 0}\frac{\bar{\varepsilon}}{\Delta\bar{r}}\bar{v}; \tag{22}$$

$$\lim_{\Delta\bar{r}\to 0}\frac{\Delta a}{\Delta\bar{r}} = \frac{d\bar{a}}{d\bar{r}};\qquad \lim_{\Delta\bar{r}\to 0}\frac{\bar{\varepsilon}}{\Delta\bar{r}}\bar{v} = 0.$$

In the left side of (22) we have obtained a certain operation on the quantities $d\bar{a}/d\bar{r}$ and $\bar{v}$.

In order to establish what the derivative $d\bar{a}/d\bar{r}$ and the entire operation represent by themselves, we will consecutively substitute into (22) the unit-vectors $\bar{i}, \bar{j}, \bar{k}$ instead of the vector $\bar{v}$ and we will obtain

$$\frac{d\bar{a}}{d\bar{r}}\bar{i} = \frac{\partial\bar{a}}{\partial x};\quad \frac{d\bar{a}}{d\bar{r}}\bar{j} = \frac{d\bar{a}}{dy};\quad \frac{d\bar{a}}{d\bar{r}}\bar{k} = \frac{\partial\bar{a}}{\partial z}.$$

In accordance with the definition of the tensor we have

$$\frac{d\bar{a}}{d\bar{r}} = \begin{Vmatrix} \frac{\partial a_x}{\partial x} & \frac{\partial a_x}{\partial y} & \frac{\partial a_x}{\partial z} \\ \frac{\partial a_y}{\partial x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial z} \\ \frac{\partial a_z}{\partial x} & \frac{\partial a_z}{\partial y} & \frac{\partial a_z}{\partial z} \end{Vmatrix}. \tag{23}$$

In this manner, "the derivative of the vector in respect to a vector" is represented by a tensor having the form (23).

We will examine the relationships between the integrals over the volume (τ) and over the surface (σ) which limits the given volume

$$\int_{\tau} (\operatorname{grad} a) \cdot \bar{b}\, d\tau = -\int_{\sigma} (a\bar{n}) \cdot \bar{b}\, d\sigma - \int_{\tau} I_1 \left(\frac{d\bar{b}}{dr} \cdot K_a \right) d\tau, \qquad (24)$$

where $\bar{n}$ is the internal normal to the surface σ.

The dot in the integrand of the second integral of the right side denotes the scalar multiplication of a tensor by another tensor, that is the multiplication of matrix by another matrix and consequently,

$$\int_{\tau} (\Delta' \bar{a}) \cdot \bar{b}\, d\tau = -\int_{\sigma} \left(\frac{d\bar{a}}{dr} \cdot \bar{n} \right) \bar{b}\, d\sigma - \int_{\tau} I_1 \left(\frac{d\bar{a}}{dr} \cdot K \frac{d\bar{b}}{dr} \right) d\tau, \qquad (25)$$

where $\bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k}$; $b = b_x \bar{i} + b_y \bar{j} + b_z \bar{k}$ - are two vectors, whose components are functions of x, y and z. $\bar{n}$ is the internal normal to the surface (σ) which limits the volume (τ); $\frac{d\bar{a}}{dr}$ и $\frac{d\bar{b}}{dr}$ are matrixes which are defined by means of the formula (23).

The Equations of Motion of a Rigid Body Around a Stationary Point for the Case When the Body Contains Cavities Filled by Viscous Incompressible Liquid

The motion of a rigid body containing a stationary point and having cavities filled with a liquid are often encountered in Nature and in Technology. As an example, it is immediately possible to point to the motion of the Earth. Similar motions play an important role in a number of fields of present day technology.

Below we derive general equations of motion of a rigid body with a liquid filling, moving around a stationary point, without establishing any limitation on the form of the cavities inside the body and on the character in which it was filled with the liquid. The method of derivative consists of the application of the general theorems of dynamics of a system of particles. Afterwards we examine a case where the body contains a cavity entirely filled with a viscous liquid.

It is possible to obtain from the introduced equations of motion by means of tensor analysis a new expression of one of the basic results of the work of N. Ye. Zhukovskiy - a theorem that characterizes the change of kinetic energy of the entire mechanical system in the process of the system's motion /1/.

The new formulation of the theorem is mathematically much more simple in comparison with the form that is presented in the original work of N. Ye. Zhukovskiy.

We will further investigate in detail one particular case of motion of a rigid body containing cavities fully filled with a liquid; this case has great significance in technological applications and consists in the fact that the rigid envelope and the liquid contained within it move as a single rigid body. A similar motion was named "limiting motion" by N. Ye. Zhukovskiy. Below there are derived the conditions for the existance of such a limiting motion.

1. We will examine a material system consisting of a rigid envelope and cavities which are partially or totally filled by a liquid; let this system move in such a manner that the point 0 which is the center of gravity of the rigid envelope remains stationary.

We will introduce two coordinates systems; a stationary one $\xi\eta\zeta$ with an origin at the point 0 and a moving one xyz unalterably coupled to the rigid envelope and with an origin at the same point 0.

We further introduce the following denotations:

I — is the tensor of inertia of the rigid part of the system in respect to 0;

$\bar{L}$ — kinetic moment of the liquid part in its own motion, that is, in motion relative to the system $0\xi\eta\zeta$;

$\bar{\omega}$ — instantaneous angular velocity of rotation of the rigid body around point 0;

$\bar{M}_e$ — is the moment of external forces acting on the entire system, relative to the point 0.

The kinetic moment of the rigid part relative to the point 0 will be equal to $I\bar{\omega}$.

We will apply to the entire given system the theorem of change of the kinetic moment relative to the point 0:

$$\frac{d}{dt}(J\bar{\omega}+\bar{L})_0=\bar{M}_e. \qquad (26)$$

We will express total derivatives of the vectors by local derivatives on moving

axis according to the Boor formulas

$$\frac{d\bar{L}}{dt} = \frac{\tilde{d}\bar{L}}{dt} + \bar{\omega} \times \bar{L}$$

and

$$\frac{d}{dt}(I\bar{\omega}) = I\frac{\tilde{d}\bar{\omega}}{dt} + \bar{\omega} \times I\bar{\omega},$$

where the symbol $\frac{\tilde{d}}{dt}$ denotes a local derivative.

We will then rewrite the equation (26) in the form

$$I\frac{\tilde{d}\bar{\omega}}{dt} + \bar{\omega} \times (I\bar{\omega} + \bar{L}) + \frac{\tilde{d}\bar{L}}{dt} = \bar{M}_e. \tag{26'}$$

This equation represents the most general initial form of equations of motion of a rigid body containing liquid-filled cavities. By making various assumptions about the liquid part, it is further possible to obtain separate particular cases of equations of motion. Let the tensor of inertia of the rigid part in the chosen moving coordinate system have the form

$$I = \begin{Vmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{Vmatrix}. \tag{27}$$

We denote p, q and r as the projections of the vector $\bar{\omega}$ of the instantaneous angular velocity of rotation of the rigid part of the system, and consequently of the axis Oxyz into the moving coordinate axes; $\bar{L}_x$, L_y and L_z are the projections of the vector $\bar{L}$ of the kinetic moment in the absolute motion of the liquid masses upon the same axis.

The moment of internal forces $\bar{M}_e$ we will represent as a sum of the components

$$\bar{M}_e = \bar{M}_1 + \bar{M}_2,$$

where $\bar{M}_1$ is the moment of forces applied to the rigid part, and $\bar{M}_2$ is the moment of the forces applied to the liquid part.

We will project the equation (26) on the moving coordinate axes. If $\bar{i}$, $\bar{j}$ and $\bar{k}$ are unit vectors of the given axes then $\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k}$:

$$I\bar{\omega} = (I_{xx}p - I_{xy}q - I_{xz}r)\bar{i} + (-I_{yx}p + I_{yy}q - I_{yz}r)\bar{j} + \\ + (-I_{zx}p - I_{zy}q + I_{zz}r)\bar{k};$$

$$I\frac{\tilde{d}\bar{\omega}}{dt} = (I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r})\bar{i} + (-I_{yx}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r})\bar{j} + \\ + (-I_{zx}\dot{p} - I_{zy}\dot{q} + I_{zz}\dot{r})\bar{k},$$

where the time derivatives are denoted by a dot. We will further write the vector product $\bar{\omega} \times (J\bar{\omega} + \bar{L})$ in terms of the moving axes.

$$\bar{\omega} \times (I\bar{\omega} + \bar{L}) =$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ I_{xx}p - I_{xy}q - I_{xz}r + L_x & -I_{yx}p + I_{yy}q - I_{yz}r + L_y & -I_{zx}p - I_{zy}q + I_{zz}r + L_z \end{vmatrix};$$

The three equations projected on the moving coordinate axes will correspond to the equation of motion (26') which was written in vectorial form. If we will direct the axes of the moving coordinate system along the major inertial axes of the rigid envelope at the point O, then the equations of motion will take the following form upon performing uncomplicated simplifications

$$\begin{aligned} I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + qL_z - rL_y + \frac{dL_x}{dt} &= M_{1x} + M_{2x}; \\ I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + rL_x - pL_z + \frac{dL_y}{dt} &= M_{1y} + M_{2y}; \\ I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + pL_y - qL_x + \frac{dL_z}{dt} &= M_{1z} + M_{2z}. \end{aligned} \quad (28$$

The equations (28) are analogous to the Euler equations of motion of a rigid body around a stationary point, but in these equations there have been added terms which characterize the influence of the motion of the liquid part on the motion of the rigid envelope, that is the components which can be divided into two groups

$$qL_z-rL_y;\ rL_x-pL_z;\ pL_y-qL_x$$

and

$$\frac{dL_x}{dt};\quad \frac{dL_y}{dt};\quad \frac{dL_z}{dt}.$$

Finally we wish to point out that the equations (28) do not depend on the degree to which the liquid fills the internal cavities.

2. We will examine the case of viscous liquid which totally fills the cavities of the body. Let the system consist of a rigid body and a viscous non-homogeneous liquid mass, contained in cavities whose number we will denote by h. Let forces $\bar{F}_i$ having a potential $U_i=U_i(x, y, z)$, that is, $\bar{F}_i$=grad U_i, act on the points $P_i(x_i, y_i, z_i)$ of the liquid masses. We will denote by $\tau_1, \tau_2, \ldots \tau_n$ — the volumes of the cavities which are filled with the liquid, and by $\bar{r}_i(x_i, y_i, z_i)$ the radius-vectors of the points of the liquid. Then the moment P_i of a force applied at any point is equal to

$$P_i=\bar{r}_i\times\bar{F}_i,$$

and the vector is

$$\bar{M}_2=\sum_1^h\int_\tau(\bar{r}_i\times\bar{F}_i)\,d\tau.$$

In further discussion we will omit the signs pertaining to the points and to the force functions in separate cavities. Since $\bar{F}$=grad U, Then $\bar{r}\times\bar{F}$=-grad U$\times\bar{r}$, consequently,

$$\bar{M}_2=\sum_1^h\int_\tau(\bar{r}\times\bar{F})\,d\tau=-\sum_1^h\int_\tau(\text{grad}\,U\times\bar{r})\,d\tau$$

or in the projections on the coordinate axes

$$M_{2x}=\sum_{1}^{h}\int_{\tau}\left(y\frac{\partial U}{\partial z}-z\frac{\partial U}{\partial y}\right)d\tau=\sum_{1}^{h}\int_{\tau}(yF_z-zF_y)\,dx\,dy\,dz;$$

$$M_{2y}=\sum_{1}^{h}\int_{\tau}\left(z\frac{\partial U}{\partial x}-x\frac{\partial U}{\partial z}\right)d\tau=\sum_{1}^{h}\int_{\tau}(zF_x-xF_z)\,dx\,dy\,dz;$$

$$M_{2z}=\sum_{1}^{h}\int_{\tau}\left(x\frac{\partial U}{\partial y}-y\frac{\partial U}{\partial x}\right)d\tau=\sum_{1}^{h}\int_{\tau}(xF_y-yF_x)\,dx\,dy\,dz.$$

We can now write the equation (26') in the form

$$J\dot{\overline{\omega}}+\overline{\omega}\times(J\overline{\omega}+\overline{L})+\frac{d\overline{L}}{dt}=\overline{M}_1-\sum_{1}^{h}\int_{\tau}(\operatorname{grad}U\times\overline{r})\,d\tau$$

or in the terms of projections on the coordinate axes:

$$I_{xx}\dot{p}+(I_{zz}-I_{yy})qr+qL_z-rL_y+\frac{dL_x}{dt}=$$

$$=M_{1x}-\sum_{1}^{h}\int_{\tau}(yF_z-zF_y)\,dx\,dy\,dz;$$

$$I_{yy}\dot{q}+(I_{xx}-I_{zz})pr+rL_x-pL_z+\frac{dL_y}{dt}=$$

$$=M_{1y}-\sum_{1}^{h}\int_{\tau}(zF_x-xF_z)\,dx\,dy\,dz;$$

$$I_{zz}\dot{r}+(I_{yy}-I_{xx})pq+pL_y-qL_x+\frac{dL_z}{dt}=$$

$$=M_{1z}-\sum_{1}^{h}\int_{\tau}(xF_y-yF_x)\,dx\,dy\,dz.$$

As can be seen from the equations (28) the last set of equations is insufficient for the full solution of the problem of motion of the entire given material system, consisting of the motion of the liquid and of the motion of the rigid envelope. The motion of the liquid is composed of motion relative to the rigid envelope and of translational motion together with the rigid envelope. In this manner, it is necessary to obtain additional equations, which would characterize the relative motion of the liquid and also equations which express boundary conditions.

We shall first establish the boundary conditions and afterwards

the conditions within the liquid.

Boundary Conditions

Since the liquids that fill the cavities are assumed to be viscous, then there exist frictional forces on the inner walls of the cavities, and these forces are directed opposite to the relative velocities at the points of contact; the value of the frictional forces depends upon the viscosity of the liquid and the nature of the walls of the cavities.

If $\bar{r}_1$ is the radius vector of the point of contact of the liquid with the wall (point P), $\dot{\bar{r}}_1$ is the vector of the relative velocity of the point P and $\bar{F}_{fr}$ is the force of friction between the liquid and the wall of the cavity, then

$$\bar{F}_{\text{тр}} = -m\dot{\bar{r}}_1, \tag{29}$$

where m is the proportionality coefficient, that is, the ratio of the value of the frictional force to the velocity of the particle; the frictional force is considered to be applied to the particle.

The first boundary condition at the rigid wall will be obtained from the following considerations: assuming only partial viscosity of the liquid it is possible to state that upon the liquid particle there will be acting a frictional force F_{fr} and an opposite force will be applied to the wall. This opposite force is a tangential one which is a component of the reaction on the liquid side of the wall which reaction we will denote by $\bar{F}_{\tau}$. If n is a normal to the surface at some point P, then the normal resultant $\bar{F}_n$ of the entire reaction

is equal to $(\bar{F}\cdot\bar{n})\,\bar{n}$, and the tangential component is given by

$$\bar{F}_\tau = \bar{F} - (\bar{F}\cdot\bar{n})\,\bar{n}, \tag{30}$$

that is, it is definable for known $\bar{F}$ and $\bar{n}$. On the other side, this force is directed opposite to the frictional force $\bar{F}_{fr}$, which gives us the boundary condition

$$\bar{F}_\tau = m\dot{\bar{r}}_1. \tag{31}$$

We will obtain the second boundary condition by considering the fact that the relative velocity of the liquid mass at the common points P is directed along the tangent to the surface at that point; that is, it is perpendicular to the normal to the point. The condition of perpendiculiarity of the vectors $\bar{r}_1$ and $\bar{n}$, that is,

$$\dot{\bar{r}}_1\cdot\bar{n} = 0 \tag{32}$$

gives the seond boundary condition.

If we will denote the projections of the relative velocity by u_1, v_1, w_1, and the directional cosines of the normal by α, β, γ, then the equation (32) can be written in the form

$$u_1\alpha + v_1\beta + w_1\gamma = 0. \tag{32'}$$

Conditions at the Interior Points of the Liquid Mass

Essentially, under this heading we should consider the ordinary equations of motion of a viscous liquid relative to a rigid envelope, that is, relative to the axes of the coordinate system Oxyz. Such equations are very well known, they are the Napier-Stokes' equation and the continuity equation; because, of this,

we bring them without deriving:

Continuity equation

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right)=0; \tag{33}$$

Napier-Stokes' equation

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}+w_1\frac{\partial u_1}{\partial z}=X_1-\frac{1}{\rho}\frac{\partial p}{\partial x}-$$
$$-\frac{\nu}{3}\frac{\partial}{\partial x}\operatorname{div}\dot{\bar{r}}_1+\nu\Delta u_1;$$
$$\frac{\partial v_1}{dt}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}+w_1\frac{\partial v_1}{\partial z}=Y_1-\frac{1}{\rho}\frac{\partial p}{\partial y}-$$
$$-\frac{\nu}{3}\frac{\partial}{\partial y}\operatorname{div}\dot{\bar{r}}_1+\nu\Delta v_1;$$
$$\frac{\partial w_1}{\partial t}+u_1\frac{\partial w_1}{\partial x}+v_1\frac{\partial w_1}{\partial y}+w_1\frac{\partial w_1}{\partial z}=$$
$$=Z_1-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\nu}{3}\frac{\partial}{\partial z}\operatorname{div}\dot{\bar{r}}_1+\nu\Delta w_1,$$

where X_1, Y_1 and Z_1 are the projections of the force $\bar{F}_1=X_1\bar{i}+Y_1\bar{j}+z_1\bar{k}$ on the axes of the moving system of coordinates, taking into consideration the inertial forces $\bar{F}_e$ and $\bar{F}_k$ which will be applied to the liquid particle during the relative motion.

The equations of relative motion of the liquid can be written in vector form in the following manner

$$\frac{d\dot{\bar{r}}_1}{dt}=\bar{F}+\bar{F}_e+\bar{F}_k-\frac{1}{\rho}\operatorname{grad}p-\frac{\nu}{3}\operatorname{grad}\operatorname{div}\dot{\bar{r}}_1+\nu\Delta'\dot{\bar{r}}_1. \tag{34}$$

Here $\bar{F}_e$ - is the translational inertial force
$\bar{F}_k$ - is the rotational inertial force (the inertial force due to the Corioliss acceleration)
$\nu=\frac{\mu}{\rho}$ - is the kinematic coefficient of viscosity.

For the further discussion it will be necessary to employ the equation of the absolute motion of the liquid, which we will write in the form

$$\frac{d\dot{\bar{r}}}{dt}=\bar{F}-\frac{1}{\rho}\operatorname{grad} p-\frac{\nu}{3}\operatorname{grad}\operatorname{div}\dot{\bar{r}}+\nu\Delta'\dot{\bar{r}}. \qquad (35)$$

Equations (28), (31), (32), (33) and (34) gives a full presentation of the motion of a rigid body, containing h cavities of any form, filled with a viscous compressible liquid, which body moves around a stationary point.

Kinetic Energy of the System

The kinetic energy of the entire given system consists of the kinetic energy of the rigid part together with the kinetic energy of the liquid that fills the cavities. The kinetic energy of the rigid part T_{rig} is equal to $T_{TB}=\frac{1}{2}\bar{\omega}\cdot J\bar{\omega}.$

If the axes of the coordinate system coincide with the major axes of inertia, then the kinetic energy of the rigid part is

$$T_{TB}=\frac{1}{2}(I_{xx}p^2+I_{yy}q^2+I_{zz}r^2). \qquad (36)$$

The kinetic energy of the liquid part during its absolute motion is equal to the sum of the kinetic energies of the separate particles of the liquid, taken over the volume of all the cavities. If m_i is the mass of an elementary particle, then its kinetic energy is

$$T_i=\frac{1}{2}m_i\dot{\bar{r}}_i^2.$$

It is possible to obtain the kinetic energy for one cavity if we will take the integral over the total volume of the cavity; for h cavities, noting that we have

$$T_{liq}=\frac{1}{2}\sum_1^h\int_\tau \dot{\bar{r}}^2\rho\, d\tau. \qquad (37)$$

In this manner the kinetic energy of the entire system will be

$$T=\frac{1}{2}\left(\sum_{1}^{h}\int_{\tau}\dot{\bar{r}}^{2}\rho\,d\tau+\bar{\omega}\cdot I\bar{\omega}\right) \qquad (38)$$

or

$$T=\frac{1}{2}\left[\sum_{1}^{h}\int_{\tau}(u^{2}+v^{2}+w^{2})\rho\,d\tau+I_{xx}p^{2}+I_{yy}q^{2}+I_{zz}r^{2}\right]. \qquad (38')$$

We will examine a case when the sum of the moments of internal forces acting on the rigid envelope and on the liquid mass is equal to zero. Then the right sides of the equations (26) and (28) are equal to zero. Let us further assume that the forces $\bar{F}(X,Y,Z)$, which act on a point of the liquid have a potential U; in addition, we will denote

$$\int\frac{dp}{\rho}=\pi. \qquad (39)$$

Then the equation (35) can be written in the following form

$$\ddot{\bar{r}}=\operatorname{grad}(\pi-U)-\frac{\nu}{\rho}\operatorname{grad}\operatorname{div}\dot{\bar{r}}+\nu\Delta'\dot{\bar{r}}. \qquad (35')$$

We will scalarly multiply (35') by $\dot{\bar{r}}_1\,(u_1\ v_1\ w_1)$ we will take the integrals over all the volumes $\tau_1, \tau_2, \ldots \tau_h$ as we will sum them over all the h cavities of the system

$$\sum_{1}^{h}\int_{\tau}\ddot{\bar{r}}\cdot\dot{\bar{r}}_{1}\,d\tau=\sum_{1}^{h}\int_{\tau}\operatorname{grad}(\pi-U)\cdot\dot{\bar{r}}_{1}\,d\tau- \qquad (40)$$
$$-\sum_{1}^{h}\int_{\tau}\frac{\nu}{\rho}\operatorname{grad}\operatorname{div}\dot{\bar{r}}\cdot\dot{\bar{r}}_{1}\,d\tau+\sum_{1}^{h}\int_{\tau}\nu\Delta'\dot{\bar{r}}\cdot\dot{\bar{r}}_{1}\,d\tau.$$

The first and second summation on the right are equal to zero. This can be proven by a transformation of the integrals over the volume into integrals over the surface (6), which limit these volumes.

Considering $\pi - U$ as a diagonal matrix with elements $\pi . U$ considering $\bar{n}$ to be an internal normal and applying formula (24), we will get

$$\sum_1^h \int_\tau \operatorname{grad}(\pi - U)\cdot \dot{\bar{r}}_1\, d\tau = -\sum_1^h \int_\sigma (\pi - U)\bar{n}\cdot \dot{\bar{r}}_1\, d\sigma - -\sum_1^h \int_\tau I_1\left[\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K(\pi - U)\right] d\tau.$$

In this equality the first summation at the right is equal to zero, and in accordance with the second boundary condition, we will transform the integrand of the second summation

$$\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K(\pi - U) = \begin{Vmatrix} \frac{\partial u_1}{\partial x}(\pi - U) & \frac{\partial u_1}{\partial y}(\pi - U) & \frac{\partial u_1}{\partial z}(\pi - U) \\ \frac{\partial v_1}{\partial x}(\pi - U) & \frac{\partial v_1}{\partial y}(\pi - U) & \frac{\partial v_1}{\partial z}(\pi - U) \\ \frac{\partial w_1}{\partial x}(\pi - U) & \frac{\partial w_1}{\partial y}(\pi - U) & \frac{\partial w_1}{\partial z}(\pi - U) \end{Vmatrix};$$

$$I_1\left[\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K(\pi - U)\right] = (\pi - U)\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right) = (\pi - U)\operatorname{div}\dot{\bar{r}}_1;$$

$$\int_\tau (\pi - U)\operatorname{div}\dot{\bar{r}}_1\, d\tau = \int_\sigma (\pi - U)\dot{\bar{r}}_1\cdot \bar{n}\, d\sigma,$$

and the last expression is equal to zero in accordance with the same condition. And so, the entire first summation in the expression (40) is equal to zero.

Applying formula (24) we will transform the second summation from the right in the expression (40)

$$-\sum_1^h \int_\tau \frac{\nu}{\rho}\operatorname{grad}\operatorname{div}\dot{\bar{r}}\cdot \dot{\bar{r}}_1\, d\tau = \sum_1^h \int_\sigma \frac{\nu}{\rho}(\operatorname{div}\dot{\bar{r}}\bar{n})\cdot \dot{\bar{r}}_1\, d\sigma + +\sum_1^h \int_\tau J_1\left[\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K\left(\frac{\nu}{\rho}\operatorname{div}\dot{\bar{r}}\right)\right] d\tau.$$

In the first sum from the right in the integrand $(\operatorname{div}\dot{\bar{r}}\,\bar{N}).\dot{\bar{r}}_1$ will have the form

$$(\operatorname{div}\dot{\bar{r}})\,\bar{n} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\alpha\bar{i} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\beta\bar{j} + +\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\gamma\bar{k} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\bar{n}.$$

Multiplying this vector scalarly by $\dot{\bar{r}}$, we will get

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)(\alpha u_1+\beta v_1+\gamma w_1);$$

the second bracket is equal to zero in accordance with the condition (32')

We will examine the integrand in the second summation thusly; the expression in the square brackets is equal to the product of two matrices:

$$d\dot{\bar{r}}_1/d\bar{r} \quad и \quad K\left(\frac{\nu}{\rho}\operatorname{div}\dot{\bar{r}}\right);$$

operation I_1 transforms this product of matrices into a scalar

$$I_1\left[\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K\left(\frac{\nu}{\rho}\operatorname{div}\dot{\bar{r}}\right)\right]=\frac{\nu}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right);$$

$$\int_\tau I_1\left[\frac{d\dot{\bar{r}}_1}{d\bar{r}}\cdot K\left(\frac{\nu}{\rho}\operatorname{div}\dot{\bar{r}}\right)\right]d\tau=\frac{\nu}{\rho}\int_\sigma\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)(u_1\alpha+v_1\beta+w_1\gamma)\,d\sigma.$$

In accordance with (32') the second bracket is equal to zero, and consequently the whole integral is equal to zero.

In this manner the formula (40) takes the form

$$\sum_1^h\int_\sigma \ddot{\bar{r}}\cdot\dot{\bar{r}}_1\,d\tau=\sum_1^h\int_\tau \nu\Delta'\dot{\bar{r}}\cdot\dot{\bar{r}}_1\,d\tau, \tag{40'}$$

where $\dot{\bar{r}}$ and $\dot{\bar{r}}_1$ are respectively the absolute and relative velocities of any point of the liquid mass, and $\ddot{\bar{r}}$ is the absolute acceleration of this point

Since $\dot{\bar{r}}=\dot{\bar{r}}_1+\bar{\omega}\times\bar{r},$ then

$$\dot{\bar{r}}_1 = \dot{\bar{r}} - \bar{\omega} \times \bar{r}. \tag{41}$$

We will substitute the expression of $\dot{\bar{r}}_1$ obtained from equation (41) instead of its value into equation (40') and we will multiply by ρ;

$$\sum_1^h \int_\tau \ddot{\bar{r}} \cdot \dot{\bar{r}}_1 \rho \, d\tau = \sum_1^h \frac{d}{dt} \left(\frac{1}{2} \int_\tau \dot{\bar{r}}^2 \rho \, d\tau \right) - \bar{\omega} \frac{d\bar{L}}{dt}. \tag{42}$$

We will now turn to the equations (26') and (28) which are the equations of motion of the entire system. We will multiply (26') scalarly by $\bar{\omega}$, or which is really the same, we will multiply the first equation (28) by p, the second by q, and the third by r, and we will add them. Then the terms containing the products of pqr and L_x, L_y and L_z will disappear. It will remain

$$I_{xx} p\dot{p} + I_{yy} q\dot{q} + I_{zz} r\dot{r} + p\frac{dL_x}{dt} + q\frac{dL_y}{dt} + r\frac{dL_z}{dt} = 0$$

or

$$\frac{1}{2}\frac{d}{dt}(I_{xx}p^2 + I_{yy}q^2 + I_{zz}r^2) = -\left(p\frac{dL_x}{dt} + q\frac{dL_y}{dt} + r\frac{dL_z}{dt}\right)$$

or

$$\frac{1}{2}\frac{d}{dt}(\bar{\omega} \cdot J\bar{\omega}) = -\bar{\omega} \cdot \frac{d\bar{L}}{dt}. \tag{43}$$

In this manner we have obtained the ratio of change of the kinetic energy of the rigid envelope. As can be seen, the change of the kinetic energy of the rigid envelope in the case when the moment of the internal forces relative to the stationary point is equal to zero is connected with the change in the kinetic moment of the liquid masses as given by equation (43). Substituting (43) into (42)

we will get

$$\sum_1^h \int_\tau \ddot{\bar{r}} \cdot \dot{\bar{r}}_1 \rho \, d\tau = \sum_1^h \frac{d}{dt} \frac{1}{2} \int_\tau \dot{\bar{r}}^2 \rho \, d\tau + \frac{1}{2} \frac{d}{dt} (\bar{\omega} \cdot J\bar{\omega}) =$$

$$= \frac{d}{dt} \frac{1}{2} \left(\sum_1^h \int_\tau \dot{\bar{r}}^2 \rho \, d\tau + \bar{\omega} \cdot J\bar{\omega} \right). \tag{44}$$

According to (38), the right side represents the change of kinetic energy of the entire system; that is, it is equal to dT/dt. Returning to (40') and keeping in mind that $\nu\rho = \mu$, we will obtain a theorem about the change of kinetic energy:

$$\frac{dT}{dt} = \sum_1^h \int_\tau \mu (\Delta' \dot{\bar{r}} \cdot \dot{\bar{r}}_1) \, d\tau \tag{45}$$

or

$$\frac{dT}{dt} = \sum_1^h \int_\tau \mu \left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) u_1 + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) v_1 + \right.$$

$$\left. + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) w_1 \right] d\tau. \tag{45'}$$

Above we have examine a viscous liquid, and we did not make any assumption about the size of the frictional forces acting on the walls of the cavities, although these forces are related to the term $\dot{\bar{r}}_1$ which enters into equation (45); that is, they are related to the relative velocity of the liquid particles by means of formula (29) and the relative velocity is related to the pressure by formula (31), which contains the coefficient m, which characterizes the magnitude of the forces of friction between the particles of the liquid and the body.

In order to obtain the N. Ye. Zhukovskiy theorem in the most general form we will take the most general form of pressure, and will relate it to the frictional forces and to the relative velocity. We will denote the pressure tensor by β. The tensor has the form

$$\beta = p - 2\mu \frac{d\dot{\bar{r}}}{d\bar{r}} - \frac{2}{3} \mu J_1 \frac{d\dot{\bar{r}}}{d\bar{r}}. \tag{46}$$

The pressure $\bar{F}$ at any point on the boundary will be defined as $\beta \bar{n}$:

$$\bar{F}-\beta\bar{n}=p\bar{n}-2\mu\left\|\begin{matrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{matrix}\right\|\bar{n}- \tag{47}$$

$$-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\bar{n},$$

and its projections on the coordinates axes can be defined by giving the directional cosines of the normal.

The tangential component of this pressure is

$$\bar{F}_\tau=\bar{F}-(\bar{F}\cdot\bar{n})\bar{n}. \tag{48}$$

We will transform the right side of the formular (45) employing the expressions for the tangential component of pressure and the formula (25):

$$\sum_1^h\int_\tau \mu\Delta'\dot{r}\cdot\dot{r}_1\,d\tau=-\sum_1^h\int_\sigma \mu\left(\frac{d\dot{r}}{d\dot{r}}\bar{n}\right)\cdot\dot{r}_1\,d\sigma-$$

$$-\sum_1^h\int_\tau \mu I_1\left(\frac{d\dot{r}_1}{d\dot{r}}\cdot K\frac{d\dot{r}}{d\dot{r}}\right)d\tau, \tag{49}$$

but from (47) we have

$$-\mu\frac{d\dot{r}}{d\dot{r}}\cdot\bar{n}=\frac{1}{2}\left(\bar{F}-p\bar{n}+\frac{2}{3}\mu I_1\frac{d\dot{r}}{d\dot{r}}\cdot\bar{n}\right),$$

because of this

$$-\int_\sigma\mu\left(\frac{d\dot{r}}{d\dot{r}}\cdot\bar{n}\right)\dot{r}_1\,d\sigma=\frac{1}{2}\int_\sigma\bar{F}\cdot\dot{r}_1\,d\sigma-\frac{1}{2}\int_\sigma p\bar{n}\cdot\dot{r}_1\,d\sigma+ \tag{50}$$

$$+\frac{1}{2}\int_\sigma\frac{2}{3}\mu\left(I_1\frac{d\dot{r}}{d\dot{r}}\right)\bar{n}\cdot\dot{r}_1\,d\sigma=\frac{1}{2}\int_\sigma\bar{F}\dot{r}_1\,d\sigma.$$

The remaining integrals are equal to zero, in accordance with the boundary condition (32). We will further represent the total pressure $\bar{F}$ as a sum of the tangential $\bar{F}_\tau$ and the normal $(\bar{F}\cdot\bar{n})\,\bar{n}$ and we will substitute in the last formula obtaining

$$\frac{1}{2}\int_\sigma\bar{F}\cdot\dot{r}_1\,d\sigma=\frac{1}{2}\int_\sigma\bar{F}_\tau\dot{r}_1\,d\sigma+\frac{1}{2}\int_\sigma(\bar{F}\cdot\bar{n})(\bar{n}\cdot\dot{r}_1)\,d\sigma=$$

$$=\frac{1}{2}\int_\sigma\bar{F}_\tau\cdot\dot{r}_1\,d\sigma=-\frac{1}{2}\int_\sigma\bar{F}_{\tau p}\cdot\dot{r}_1\,d\sigma.$$

Substituting all this into (45') we will get the formula we have been looking for:

$$\frac{dT}{dt} = -\frac{1}{2}\sum_{1}^{h}\int_{\sigma}\overline{F}_{\tau p}\cdot\dot{\overline{r}}_1\,d\sigma - \sum_{1}^{h}\int_{\sigma}\mu I_1\left(\frac{d\dot{\overline{r}}_1}{d\overline{r}}\cdot K\frac{d\dot{\overline{r}}}{d\overline{r}}\right)d\tau. \qquad (51)$$

In comparison with the N. Ye. Zhukovskiy formula this formula is more general, and it is also correct firstly for any number of cavities filled with a viscous fluid and secondly for any given character of the frictional forces acting at the point of contact of the liquid with the boundaries of the cavities. At the same time the formula (51) is considerably more simple than the respective N. Ye. Zhukovskiy formula and consequently it is more convenient for the calculation of the dissipation of the kinetic energy of the entire system.

Limiting Motion and Its Properties

According to the definition given by N. Ye. Zhukovskiy, a limiting motion will be a motion for which the change of kinetic energy will be equal to zero, that is dT/dt-0. We will investigate what are the conditions when the change of kinetic energy can take place. We will turn to formula (51). It was obtained for the most general assumptions in respect to the properties of the liquid and the reactive forces acting upon any particle P of the liquid by the remaining mass of the liquid. In accordance with (51) the change of kinetic energy will be equal to zero, if both the right side summations will be equal to zero, and they are, as is known, equal to zero when

$$\int_{\sigma}\overline{F}_{\tau p}\cdot\dot{\overline{r}}_1 d\sigma = 0; \quad \int_{\tau}\mu I_1\left(\frac{d\dot{\overline{r}}_1}{d\overline{r}}\cdot K\frac{d\dot{\overline{r}}}{d\overline{r}}\right)d\tau = 0.$$

We see from the second expression that it can be equal to zero if $d\dot{\overline{r}}/dr=0$. This

means that the relative velocity of the point is independent from the position of the point, that is, in any given point the relative velocity will be the same. The first integral is equal to zero, when the scalar product of the vector of frictional forces and of the relative velocity is equal to zero, that is when one of the multiplying vectors is equal to zero, or if these vectors are mutually perpendicular. But $\bar{F}_{fr} \neq 0$ and the vector F_{fr} is parallel to the vector $\dot{\bar{r}}_1$. Because of this the relative velocity $\dot{\bar{r}}_1$ should be equal to zero. However, if the relative velocity $\dot{\bar{r}}_1$ of the liquid particles is equal to zero at the boundary and according to the first condition, this velocity is the same at all points, then it will be equal to zero at any point of the liquid mass. This means that the particles are not displaced relative to the rigid envelope, and the liquid will be moving as a single body with the rigid envelope, In this manner, the limiting motion exists then and only then when the liquid moves as a single body with the rigid envelope.

For further investigation we will take the equation (35') of motion of liquid particle

$$\ddot{\bar{r}} = \mathrm{grad}(\pi - U) + \frac{5}{3}\frac{\mu}{\rho}\,\mathrm{grad\,div}\,\dot{\bar{r}} + \nu\Delta'\dot{\bar{r}}. \qquad (35'')$$

The absolute acceleration of the particle is

$$\ddot{\bar{r}} = \frac{d}{dt}(\dot{\bar{r}}_1 + \bar{\omega} \times \bar{r}).$$

In the case of limiting motion $\dot{\bar{r}}_1 = 0$, and because of this the absolute acceleration of the particle is

$$\ddot{\bar{r}} = \frac{d}{dt}(\bar{\omega} \times \bar{r}).$$

We will find the time derivative of the vector product, keeping in mind that the case of limiting motion $\frac{d\bar{r}}{dt}=\bar{\omega}\times\bar{r}$ and opening up the double vector product we have

$$\frac{d}{dt}(\bar{\omega}\times\bar{r})=\frac{d\bar{\omega}}{dt}\times\bar{r}+(\bar{\omega}\cdot\bar{r})\bar{\omega}-\omega^2\bar{r};$$

$$\dot{\bar{r}}=\bar{\omega}\times\bar{r}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ p & q & r\\ x & y & z\end{vmatrix}=\bar{i}(qz-ry)+\bar{j}(rx-pz)+\bar{k}(py-qx);$$

$$\operatorname{div}\dot{\bar{r}}=\frac{\partial}{\partial x}(qz-ry)+\frac{\partial}{\partial y}(rx-pz)+\frac{\partial}{\partial z}(py-qx).$$

This expression, as can be easily seen, is equal to zero.

Further, we have by definition

$$\begin{aligned}\Delta'\dot{\bar{r}}=&\left[\frac{\partial^2}{\partial x^2}(qz-ry)+\frac{\partial^2}{\partial y^2}(qz-ry)+\frac{\partial^2}{\partial z^2}(qz-ry)\right]\bar{i}+\\&+\left[\frac{\partial^2}{\partial x^2}(rx-pz)+\frac{\partial^2}{\partial y^2}(rx-pz)+\frac{\partial^2}{\partial z^2}(rx-pz)\right]\bar{j}+\\&+\left[\frac{\partial^2}{\partial x^2}(py-qx)+\frac{\partial^2}{\partial y^2}(py-qx)+\frac{\partial^2}{\partial z^2}(py-qx)\right]\bar{k}=\\&=\left(-\frac{\partial r}{\partial y}+\frac{\partial q}{\partial z}\right)\bar{i}+\left(\frac{\partial r}{\partial x}-\frac{\partial p}{\partial z}\right)\bar{j}+\left(-\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\right)\bar{k}.\end{aligned}$$

Each one of the derivatives is equal to zero and because of this $\Delta'\dot{\bar{r}}$ is also equal to zero. In this manner, it is possible to rewrite (35") in the form

$$\frac{d\bar{\omega}}{dt}\times\bar{r}+(\bar{\omega}\cdot\bar{r})\bar{\omega}-\omega^2\bar{r}=\operatorname{grad}(\pi-U). \qquad (35''')$$

In this expression the right side is a function of x, y and z, and the angular velocity $\bar{\omega}$ is the same for any point, which means that the time derivative of the angular velocity is independent of the choice of a point. It can not be a function of time since the right side does not contain time. We will further assume that this derivative is constant, that is $d\omega/dt$-const, then $\bar{\omega}$ will be a function of time. This is impossible, since the right side does not contain t. This means that $d\bar{\omega}/dt$ can only be zero, and this in turn means that $\bar{\omega}=\bar{\omega}_0$ that is

the angular velocity is a constant vector fixed in space. Consequently, limiting motion is a uniform rotation around an axis fixed in space.

We will further examine separately the change of kinetic energy of the rigid and liquid parts.

For the liquid parts we have from equation (37)

$$\frac{dT_{liq}}{dt} = \sum_1^h \frac{d}{dt}\frac{1}{2}\int_{\tau_k} \dot{\bar{r}}^2 \rho\, d\tau = \sum_1^h \int_\tau \ddot{\bar{r}}\cdot\dot{\bar{r}}_1 \rho\, d\tau + \bar{\omega}\cdot\frac{d\bar{L}}{dt}.$$

In the case of limiting motion the first summation from the right is equal to zero, since $\dot{\bar{r}}_1 = 0$. Because of this we have

$$\frac{dT_{rig}}{dt} = \bar{\omega}\cdot\frac{d\bar{L}}{dt}. \tag{52}$$

Earlier we have seen that is the moment of internal forces relative to the stationary point is equal to zero, then the change of kinetic energy of the rigid part is determined by the formular (43)

$$\frac{dT_{rig}}{dt} = -\bar{\omega}\cdot\frac{d\bar{L}}{dt}.$$

if we will consider a rigid body rotating with an angular velocity ω, then for this body the change of kinetic energy is

$$\frac{dT_{rig}}{dt} = J\bar{\omega}\cdot\frac{d\bar{\omega}}{dt}.$$

But since in the case of limiting motion $\bar{\omega} = \bar{\omega}_0 = \text{const}$, then $d\omega/dt = 0$ and for $\omega_0 \neq 0$ $\frac{dT_{TB}}{dt} = -\bar{\omega}_0\cdot\frac{d\bar{L}}{dt}$ can only be equal to zero in case $d\bar{L}/dt = 0$,

and consequently also $dT_{liq}/dt=0$ that is neither for the rigid envelope nor for the filling liquid, taken separately does there occur a change in the kinetic energy.

What kind of a body and its liquid filling should there be in order to make sure that the motion will be a limiting one?

We have established that in the case of limiting motion the derivative of the kinetic moment $\bar{L}$ of the liquid masses in respect to the time t is equal to zero and the velocity $\bar{\omega}$ of motion of the body is constant in respect to time and space. because of this the equations (26) and (26') simplify and take a form

$$\bar{\omega}\times[I\bar{\omega}\times\bar{L}]=0. \tag{53}$$

It has been remarked earlier that in the case of limiting motion the liquid masses move together as a single entity with the rigid envelope; that is, it is possible to investigate a system consisting of the rigid part and the liquid-filled part as one absolutely rigid body. We will introduce the tensor of inertia I_0 for the given body, when the kinetic moment of the system for the angular velocity $\bar{\omega}$ will be equal to $I\bar{\omega}$, that is

$$I\bar{\omega}+\bar{L}=I_0\bar{\omega}. \tag{54}$$

This expression can be equal to zero in such a case that each coefficient of the unit vectors $\bar{i}, \bar{j}, \bar{k}$, is equal to zero; that is, the vector equation (53) is equivalent to the three scalar equations

$$\left.\begin{aligned}(\overset{\circ}{I}_{zz}-\overset{\circ}{I}_{yy})\,qr&=0,\\(\overset{\circ}{I}_{xx}-\overset{\circ}{I}_{zz})\,pr&=0,\\(\overset{\circ}{I}_{yy}-\overset{\circ}{I}_{xx})\,qp&=0.\end{aligned}\right\} \tag{53'}$$

These conditions are satisfied in the following cases:

1. If the ellipsoid of inertia of the whole system relative to the point O is a sphere, then the moments of inertia relative to all the three axes are mutually equal; that is $P_{xx} = P_{yy} = P_{zz}$. Limiting motion can be the rotation around any axis passing through the center of this sphere.

2. If the ellipsoid of inertia of the system relative to the point O is an ellipsoid of revolution (for instance, around the axis z), then the two moments of inertia $P_{xx} = P_{yy}$, are mutually equal, and in order to satisfy the equations (53') it is necessary that either p or q should bt the same time be equal to zero; meaning that r=0.

The first of these conditions signifies that the axis of revolution of the body should coincide with the axis of revolution of the ellipsoid of inertia, that is, with the axis of dynamical symmetry; in this case the motion of the body will be limiting.

The second condition (r=0) signifies that the axis of rotation of the body should lie in a plane perpendicular to the axis of rotation of the ellipsoid of inertia; in this case limiting motion will be the rotation of a body around any axis lying in a plane perpendicular to the axis of rotation of the ellipsoid of inertia.

3. If the ellipsoid of inertia, has different axis in respect to the point, that is, if $P_{xx} \neq P_{yy} \neq P_{zz}$, then, in order to satisfy (53'), it is necessary that two of the projections of the angular velocity of body's rotation on the

coordinate system should be equal to zero, and this is possible when the axis of rotation of the body will coincide with one of the major axes of the ellipsoid of inertia. For such a system the limiting motion will be rotation around the axis coinciding with either of the major axes of the ellipsoid of inertia.

This rotation takes place with constant angular velocity, and the axis of rotation remains stationary in space.

All the conclusions that were obtained can be useful in ballistics in the investigation of motions of projectiles with a liquid filling.

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VIBRATIONS OF VISCOUS LIQUIDS CONTAINED IN A CYLINDRICALLY SHAPED CAVITY, WHICH IS IN A STATE OF HORIZONTAL MOTION

By

I. S. Chikina

The problem of motion of liquid-filled body and of the motion of liquids in cavities of various forms is now of great interest.

We will examine a cylinder with its axis in horizontal position, filled with a viscous liquid which has a free surface. We will assume that the center of gravity of the cylinder moves horizontally with a constant acceleration a.

We will make the following assumptions:

1) the motion is planar, that is, the motion of the liquid particles in the vertical planes parallel to the axis of the cylinder is uniform,

2) the acceleration of the liquid particles in the direction perpendicular to the axis of the cylinder is negligeably small,

3) the deflections of the liquid particles from the free undisturbed surface are small, so that it is possible to disregard terms of the second order of smallness.

We will choose a moving set of coordinate axes with an origin at the center of gravity of the cylinder; we will direct the x-axis u upwards, the z-axis along the axis of the cylinder, and the y-axis perpendicualr to the axis of the cylinder.

The volume) forces acting on the liquid particles have a potential

$$U = az + gx + \text{const.}$$

The hydrodynamic equations of a viscous fluid are written in the general case in the form

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\\
&=X-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\\
&\quad+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right);\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\\
&=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\nu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\\
&\quad+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right);\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\\
&=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\nu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\\
&\quad+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),
\end{aligned}\right| \qquad (1)$$

where u, v, w — are the projections of the relative velocity of the particles on the chosen moving coordinate axes

p — is the pressure at any given point M (x, y, z) of the liquid

X, Y, Z — are the projections of the forces acting on the particles of the fluid, projected on the same set of axes.

In accordance with the assumptions which we have made, the equations (1) simplify to

$$0=-g-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right);$$
$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y};$$
$$\frac{\partial w}{\partial t}=-a-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\nu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right), \quad (2)$$

The following conditions are satisfied at the internal points of the liquid:

1) the continuity equation, which, for the assumptions we have made, has the form

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0; \quad (3)$$

2) pressure p at some point M, located at the distance x from the origin of coordinates

$$p=p_0+\rho\partial(h+\xi-x), \quad (4)$$

where p_0 – is the pressure above the liquid surface,
h – is the initial level of the indisturbed liquid,
ξ – is the vertical deflection of the liquid particles from their initial position, which they had in the undisturbed state.

Boundary Conditions at the walls of the cavity the velocities of the liquid particles are equal to zero due to adherence of the particles to the walls:

1) on the side walls of the cylinder having a length 2l

$$w=0 \text{ при } z=\pm l; \quad (5)$$

2) on the bottom of the cylinder u=0.

Initial Conditions

1) deflection of the liquid particles from the initial level

$$\xi = 0 \text{ at } t = 0 \qquad (6)$$

2) projections of the initial velocities on the coordinate axes

$$\dot{\xi} = 0 \text{ at } t = 0; \qquad (6')$$

For points situated on the free surface, the projections of velocity on the x-axis are equal to the derivative of the deflection

$$u = \dot{\xi} = \frac{\partial \xi}{\partial t}. \qquad (7)$$

The deflection ξ is independent of x and y; that is, it is a function of the z coordinate and of the time t.

The continuity equation (3) permits us to establish the dependency

$$u = -h^* \frac{\partial w}{\partial z},$$

where h* is the adjusted depth, equal to the area of the liquid on a section perpendicualr to the z-axis, divided by the bisecting chord.

Keeping in mind (7), it is possible to write the preceeding relationship in the form

$$\frac{\partial \xi}{\partial t} = -h^* \frac{\partial w}{\partial z}. \qquad (8)$$

Since we are seeking ξ in the form of a function of z and t, then w will also be a function of z and t. Because of this the third equation (2) will take the form

$$\frac{\partial w}{\partial t} = -a - g\frac{\partial \xi}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}. \qquad (9)$$

Differentiating (8) twice in respect to z, and once in respect to t, and differentiating equation (9) in respect to z, we will get, respectively

(8')

$$\frac{\partial^3 \xi}{\partial t \, \partial z^2} = -h^* \frac{\partial^3 w}{\partial z^3};$$

$$\frac{\partial^2 \xi}{\partial t^2} = -h^* \frac{\partial^2 w}{\partial z \, \partial t};$$

(8")

$$\frac{\partial^2 w}{\partial t \, \partial z} = -g \frac{\partial^2 \xi}{\partial z^2} + \nu \frac{\partial^3 w}{\partial z^3}.$$

(9')

Substituting into equation (9') instead of $\frac{\partial^2 w}{\partial t \partial z}$ and $\frac{\partial^3 w}{\partial z^3}$ their expressions given by (8') and (8"), we will obtain an equation for the determination of ξ (z,t)

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial z^2} + \nu \frac{\partial^3 \xi}{\partial t \, \partial z^2}, \tag{10}$$

where $c^2 = gh^*$.

We are looking for a solution of equation (10) in the form of the sum

$$\xi(z, t) = \xi_1(z) + \xi_2(z, t),$$

where $\xi_1(z)$ is a particular solution which must satisfy equation (9) and the boundary condition (5); $\xi_2(z,t)$ we wish to find in the form of a derivative of the functions $Z(z) \cdot T(t)$ each of which is a function of only one variable.

Upon substituting into equation (10) the derivatives of ξ_2 in respect to the respective variables (arguments) we will separate the variables

$$\frac{Z''}{Z} = \frac{T''}{\nu T' + c^2 T}.$$

Denoting these expressions by $-\lambda^2$, we will obtain two ordinary differential equations with constant coefficients:

$$Z'' + \lambda^2 Z = 0. \tag{11}$$

$$T'' + \nu\lambda^2 T' + c^2 \lambda^2 T = 0. \tag{12}$$

The general solution of the equation (11) will be

$$Z = C_1 \cos \lambda z + C_2 \sin \lambda z.$$

Equation (12) is a differential equation of damped vibrations. The damping is characterized by the second component which contains the coefficient of viscosity of the liquid. The general solution of the equation (12) has the form

$$T = C_3 e^{(-n+\sqrt{n^2-k^2})t} + C_4 e^{(-n-\sqrt{n^2-k^2})t},$$

where

$$2n = \nu\lambda^2;\ k = c\lambda \text{ или } k^2 = gh^*\lambda^2.$$

For a viscous fluid $n > k$ and the motion will be non-periodic.

For liquid having a relatively small viscosity, that is for $n < k$, the solution can be written in the form

$$T = e^{-nt}\left(C_5 \cos \sqrt{k^2-n^2}\,t + C_6 \sin \sqrt{k^2-n^2}\,t\right). \tag{13}$$

We further find the particular solution $\xi(z)$ by means of equation (9). The term containing $\frac{\partial^2 w}{\partial z^2}$ disappears since

$$\frac{\partial^2 w}{\partial z^2} = -\frac{1}{h^*}\frac{\partial u}{\partial z} = -\frac{1}{h^*}\frac{\partial^2 \xi}{\partial z\,\partial t},$$

and since ξ_1 is independent of t, then $\frac{\partial^2 w}{\partial z^2}$ which corresponds to

the solution $\xi_1(z)$ is equal to zero.

On integrating (9) we will get

$$w = -at - g\frac{\partial \xi_1}{\partial z}t + \psi(z); \tag{14}$$

and at t=0 ω=0, and because of this $\Psi(z)=0$.

On the basis of the boundary condition (5), which holds true for any instant of time, we get

$$\xi_1 = -\frac{a}{g}z. \tag{15}$$

The solution of equation (10) can be presented in the form

$$\xi = -\frac{a}{g}z + e^{-nt}(C_1\cos\lambda z + C_2\sin\lambda z)(C_5\cos\sqrt{k^2-n^2}\,t + \\ + C_6\sin\sqrt{k^2-n^2}\,t). \tag{16}$$

We will equate one of the arbitrary constants to zero, and we will find the other arbitrary constants and the value of λ from the initial and boundary conditions.

Let $C_1=0;\ C_2\cdot C_5=A;\ C_2\cdot C_6=B,$ then $\xi = -\frac{a}{g}z + e^{-nt}(A\cos\sqrt{k^2-n^2}\,t +$

$$+ B\sin\sqrt{k^2-n^2}\,t)\sin\lambda z. \tag{16'}$$

Differentiating above in respect to time, substituting into (8) and integrating, we will get

$$W = -\frac{1}{h^*\lambda}e^{-nt}\left[(An - B\sqrt{k^2-n^2})\cos\sqrt{k^2-n^2}\,t + \\ + (Bn + A\sqrt{k^2-n^2})\sin\sqrt{k^2-n^2}\,t\right]\cos\lambda z.$$

For $z=\pm 1$, ω =0 at any instant of time, which is only possible for

$\cos \lambda z = 0$ that is at

$$\lambda l = \frac{2N+1}{2}\pi \quad \text{или} \quad \lambda = \frac{2N+1}{2l}\pi \qquad (N=0;\ 1;\ 2, \ldots).$$

Since λ is dependent on N, we will first denote

$$\lambda_N = \frac{2N+1}{2l}\pi. \tag{17}$$

For the arbitrary constants A and B we will get and infinite number of values; as a result of this, ξ can be presented in the form of a summation ξ (z) and an infinite series

$$\xi = -\frac{a}{g}z + \sum_{N=0}^{\infty} e^{-n_N t}\left(A_N \cos\sqrt{k_N^2 - n_N^2}\,t + B_N \sin\sqrt{k_N^2 - n_N^2}\,t\right)\sin\lambda_N z.$$

The constants A_N and B_N we will find from the initial conditions that $\xi = 0$ at t=0, that is

$$0 = -\frac{a}{g}z + \sum_{N=0}^{\infty} A_N \sin\lambda_N z; \tag{18}$$

and $\dot{\xi} = 0$ at t=0, that is

$$\dot{\xi} = 0 \text{ при } t = 0, \text{ т. е.} \tag{18'}$$

$$0 = -n\sum_{N=0}^{\infty} A_N \sin\lambda_N z + \sqrt{k^2 - n^2}\sum_{N=0}^{\infty} B_N \sin\lambda_N z.$$

Multiplying both parts of the equations (18) and (18') by $\sin\lambda_m$ zdz and integrating between the limits of -l to +l, we will get

$$A_N = (-1)^N \frac{2a}{g\lambda_N^2 l}; \quad B_N = (-1)^N \frac{2an}{g\lambda_N^2 l\sqrt{k^2 - n^2}}.$$

The form of the free surface of the liquid is characterized by the equation

$$\xi = -\frac{a}{g}z + \frac{2a}{gl}\sum_{N=0}^{\infty}\frac{(-1)^N}{\lambda_N^2} e^{-nt}\left(\cos\sqrt{k^2 - n^2}\,t + \frac{n}{\sqrt{k^2 - n^2}}\sin\sqrt{k^2 - n^2}\,t\right)\sin\lambda_N z \tag{19}$$

or

$$\xi = -\frac{a}{g} z + \frac{2a}{gl} \sum_{N=0}^{\infty} \frac{(-1)^N}{\lambda_N^2} e^{-nt} H_N \times \times \cos\left(\sqrt{k^2 - n^2}\, t - \alpha_N\right) \sin \lambda_N z, \tag{19'}$$

where

$$H_N = \frac{k}{\sqrt{k^2 - n^2}}; \quad \operatorname{tg} \alpha_N = \frac{n}{\sqrt{k^2 - n^2}}.$$

P. I. Gor'kov has determined in /1/ that the form of the free surface of an ideal liquid in a disturbed state is characterized by the equation:

$$\xi = -\frac{a}{g} z + \frac{2a}{gl} \sum_{N=0}^{\infty} \frac{(-1)^N}{\lambda_N^2} \cos c\lambda_N t \sin \lambda_N z.$$

If we will set the coefficient of viscosity in equation (19) or (19') equal to zero, we will obtain the same result. It is not difficult to see the period of vibrations of the viscous liquid is equal to

$$\frac{2\pi}{\sqrt{k^2 - n^2}},$$

that is, it will be greater than for an ideal fluid; also, the waves impinging upon the surface $\xi = -\frac{a}{g} z$, are displaced in the firection of motion of the body in comparison with the waves in an ideal liquid.

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DETERMINATION OF THE PERIOD OF VIBRATIONS OF A RESTRICTED RELEASING REGULATOR

By

N. K. Lobacheva

A restricted releasing regulator consists of a vibrating balancing system - a straight spring mated with a toothed wheel.

T, the period of vibrations of the balance, is one of the basic characteristics of the releasing regulator. Considering the regulator to be a system with self-induced vibrations it is possible to obtain a formula which will permit

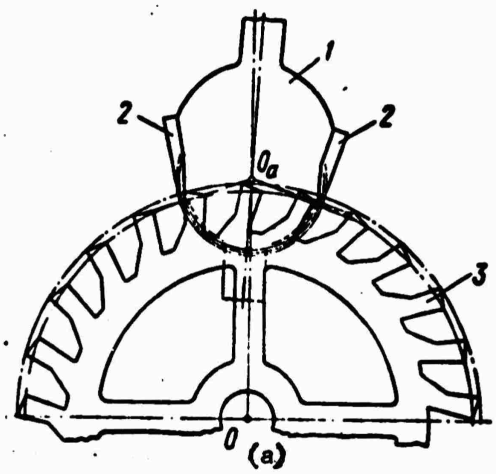

Fig. 1. Movement of a restricted release regulator. 1) balance; 2) blades; 3) runner. a) runner (abbreviation)

us to determine the period of vibrations depending upon the parameters of the regulator.

During the motion from the extreme right to the extreme left position, the balance traverses the following angular sections (positive direction is counterclockwise):

γ_B — is the angle of displacement on the input blades (from $\varphi = \Phi$ to $\varphi = \varphi_1$)

λ — is the impulse angle on the input blade (from $\varphi = \varphi_1$ to $\varphi = -\varphi_2$)

∂ — is the angle of free rotation (from $\varphi = -\varphi_2$ to $\varphi = -\varphi_3$)

γ_T — is the braking angle (from $\varphi = -\varphi_3$ to $\varphi = -\Phi$).

We will examine stable periodical motions of the balance, having a constant amplitude. Because of this we will assume that the motion of the balance from the right to the left is identical to the motion from left to right.

Using the systems presented in figures 2 and 3, and considering only the basic disturbances acting on the balance, we will set up the differential equation of motion of the balance on the parts of path enumerated below.

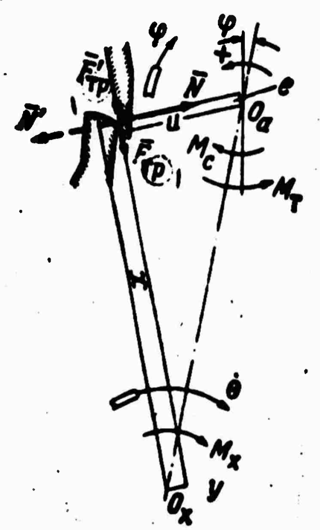

Figure 2
The tooth of the runner and the output blade during the time when the balance traverses the displacement angle.
1) friction (subscript)

$\bar{F}_{TP} = \bar{F}_{friction} = \bar{F}_{fr.}$

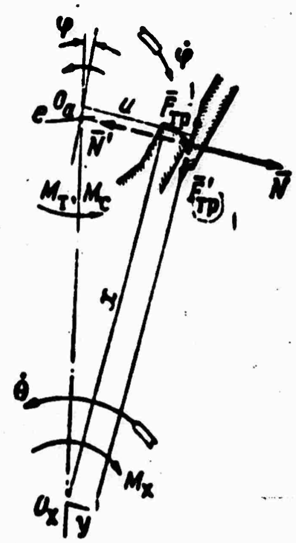

Figure 3
The tooth of the runner and the output blade in the time when the balance traverses the braking angle.
1) friction (subscrcipt)

For the angle of displacement and impluse

$$J_6\ddot{\varphi} = -M_0\varphi + \mu\varphi^3 \pm M_\tau \mp (M_x + J_x\ddot{\theta})\frac{e - fu}{x + fy}; \tag{1}$$

For the angles of free rotation

$$J_6\ddot{\varphi} = -M_0\varphi + \mu\varphi^3 \pm M_\tau; \tag{2}$$

Striking of the blades by the runner

$$J_6\ddot{\varphi} = M_y(\varphi, \dot{\varphi}); \tag{3}$$

For the braking angles

$$J_6\ddot{\varphi} = -M_0\varphi + \mu\varphi^3 \pm M_\tau \pm (M_x + J_x\ddot{\theta})\frac{e + fu}{x - fy}. \tag{4}$$

Here J_6 and J_x — are the moments of inertia of the balance and the runner,

$\ddot{\varphi}, \ddot{\theta}$ — are the angular accelerations of the balance and the runner,

M_0 — is the tensile stiffness of the spring,

μ — is the proportionality coefficient

M_τ — is a constant moment which allows for the friction in the supports of the balance,

M_x — is the turning moment on the axis of the runner,

f — is the coefficient of sliding friction of the tooth of the runner on the blade,

$M_y(\varphi, \dot{\varphi})$ — is the major moment of the forces with which the tooth of the runner acts on the balance during the time it strikes the latter.

The tope subscripts in the equations (1), (2) and (4) correspond to the motion of the balance from the left to the right and the bottom subscripts corrspond to the motion from right to left.

We will introduce functions f (φ) and f_1 (φ) which are determined by the relationships

$$f(\varphi)=\begin{cases} \dfrac{e-fu}{x+fy} & (\Phi \geqslant \varphi \geqslant -\varphi_2; \quad \dot{\varphi}<0) \\ 0 & (-\varphi_2 > \varphi > -\varphi_3; \quad \dot{\varphi}<0) \\ -\dfrac{e+fu}{x-fy} & (-\varphi_3 > \varphi > -\Phi; \quad \dot{\varphi}<0) \\ -\dfrac{e-fu}{x+fy} & (-\Phi < \varphi < \varphi_5; \quad \dot{\varphi}>0) \\ 0 & (\varphi_5 \leqslant \varphi < \varphi_6; \quad \dot{\varphi}>0) \\ \dfrac{e+fu}{x-fy} & (\varphi_6 \leqslant \varphi \leqslant \Phi; \quad \dot{\varphi}>0) \end{cases} \tag{5}$$

$$f_1(\varphi)=\frac{\dot{\theta}}{\dot{\varphi}}. \tag{6}$$

It has been shown in reference /4/ that

$$|f_1(\varphi)|=\frac{e}{x}, \tag{7}$$

and the sign of the function f_1 (φ) depends on the signs of the angular velocities of the balance and the runner.

Differentiating expression (6) in respect to time we will get:

$$\ddot{\theta}=\ddot{\varphi}f_1(\varphi)+\dot{\varphi}\dot{f}_1(\varphi). \tag{8}$$

Taking into consideration the relationships (1), (2), (3), (4) (5) and (8) we can write the differential equation of motion of the balance in the form

$$J_6\ddot{\varphi}=-M_6\varphi+\mu\dot{\varphi}^2+M_\tau(\varphi)-M_2f(\varphi)-J_2[\ddot{\varphi}F_1(\varphi)+F_2(\varphi)], \tag{9}$$

$$F_1(\varphi) = f_1(\varphi) f(\varphi); \tag{10}$$

$$F_2(\varphi) = \dot{\varphi} \dot{f}_1(\varphi) f(\varphi); \tag{11}$$

$$M_T(\varphi) = \begin{cases} M_T & (\Phi \geqslant \varphi \geqslant -\Phi; \ \dot{\varphi} < 0), \\ -M_T & (-\Phi \leqslant \varphi \leqslant \Phi; \ \dot{\varphi} > 0). \end{cases} \tag{12}$$

Figures 4 and 5 present graphs of the functions $f(\varphi)$ and $f_1(\varphi)$; the distances e, u, x and y which enter into the equations (5), (6) and (7), have been determined from drawings of the mechanism which have been constructed for different values of the angle φ. We

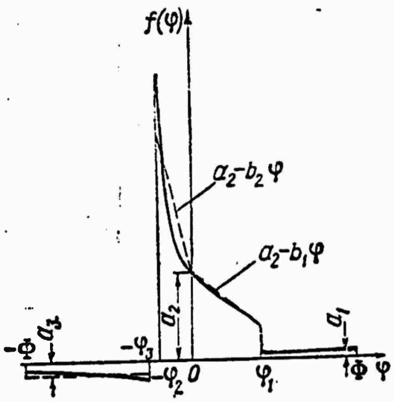

Figure 4
Graph of the function $f(\varphi)$

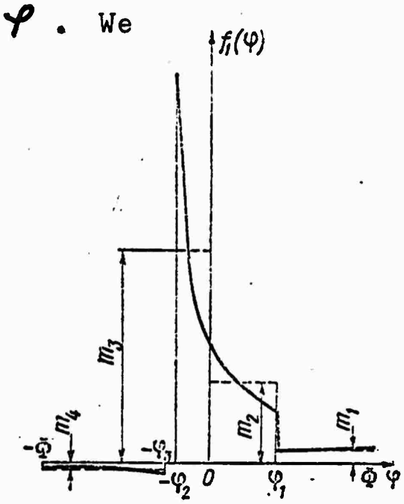

Figure 5
Graph of the function $f_1(\varphi)$

will consider the curves to be straight lines in the manner in which this is presented by the dotted lines in figures 4 and 5 and at the same time we will substitute for the function $f_1(\varphi)$ of the angle of impluses its mean values m_2 and m_3. It is now possible to represent the functions $f(\varphi)$ and $f_1(\varphi)$ by the following analytic formulas:

$$f(\varphi) = a - b\varphi, \tag{13}$$

$$f_1(\varphi) = m. \tag{14}$$

In this case $F_2(\varphi) = 0$.

Figure 6 presents a graph of the function $F_1(\varphi)$ which has been constructed in accordance with (10), (13), and (14).

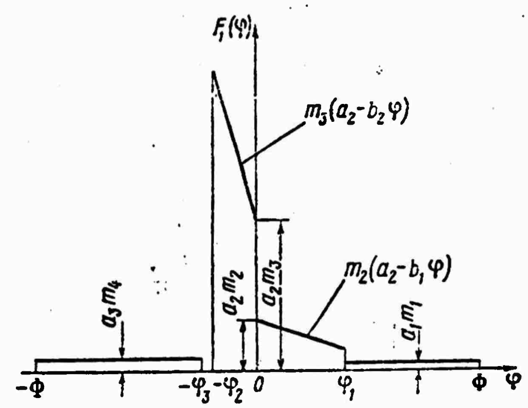

Figure 6
Graph of the function $F_1(\varphi)$

Substituting for $F_1(\varphi)$ its mean value β and introducing the symbols

$$\left.\begin{aligned} J_1 = J_6 + J_x\beta, \quad \omega^2 = \frac{M_0}{J_1}, \quad \mu_1 = \frac{\mu}{J_1}, \\ M_1(\varphi) = \frac{M_T(\varphi)}{J_1}, \quad M_2 = \frac{M_x}{J_1}, \quad M_3(\varphi, \dot{\varphi}) = \frac{M_y(\varphi, \dot{\varphi})}{J_1}. \end{aligned}\right\} \qquad (15)$$

we will write equation (9) in the form

$$\ddot{\varphi} + \omega^2\varphi - \mu_1\varphi^3 = M_1(\varphi) - M_2 f(\varphi) + M_3(\varphi, \dot{\varphi}). \qquad (16)$$

Since the balance performs vibrations which are similar to sinusoidal ones, we will assume in the first approximation

$$\varphi = \Phi \cos kt, \quad \dot{\varphi} = \mp k\sqrt{\Phi^2 - \varphi^2}. \qquad (17)$$

Here k and Φ are the frequency and amplitude of the uniform

vibrations of the system consisting of a balance and a straight spring. Since φ and $\dot{\varphi}$ are functions of time in the relationships (17) with a period $T=2\pi/k$, therefore, the functiosn which are written in the right side of the equation (16) will be periodical functiosn of time having the same period.

Letting

$$M_1(\varphi)=M_1^*(t), \quad f(\varphi)=f^*(t), \quad M_3(\varphi, \dot{\varphi})=M_3^*(t), \tag{18}$$

we have

$$\ddot{\varphi}+\omega^2\varphi-\mu_1\varphi^3=M_1^*(t)-M_2 f^*(t)+M_3^*(t). \tag{19}$$

The non-linear functions M_1^* (t) and f^* (t) (fig. 7) and also M_3^* (t) which satisfy the Dirichlet's conditions, we will expand in a Fourrier series, and in doing so, we will retain only the first terms; that is, we will present each of the mentioned functions in the form

$$y(t)=a_0^*+a_1^*\cos kt+b_1^*\sin kt.$$

After some simple computations the expansions of the functions M_1^* (t) and $f^*(t)$ into Fourrier series will take the form

(See page 238a for equation)

We will find the Fourrier coefficients for the function $M_y(\varphi, \dot{\varphi})$ using the method of computation of the influence of instantaneous impulses which is presented in the paper written by N. N. Krylov and N. N. Bogolyubov.

$$M_1^*(t)=\frac{4M_т}{\pi J_1}\sin kt,$$

$$f^*(t)=\frac{1}{\pi}\left\{\left[2(a_1-a_2)+b_1\varphi_1\right]\sqrt{1-\frac{\varphi_1^2}{\Phi^2}}+(2a_2+b_2\varphi_2)\times\sqrt{1-\frac{\varphi_2^2}{\Phi^2}}+\right.$$

$$+2a_3\sqrt{1-\frac{\varphi_3^2}{\Phi^2}}+\Phi\left(b_1\arccos\frac{\varphi_1}{\Phi}-b_2\arccos\left(-\frac{\varphi_2}{\Phi}\right)\right]+$$

$$\left.+\frac{1}{2}\pi\Phi(b_2-b_1)\right\}\cos kt+\frac{2}{\pi}\left\{a_1-a_3-\frac{1}{\Phi}\left[\varphi_1(a_1-a_2)-\right.\right.$$

$$\left.\left.-a_2\varphi_2-a_3\varphi_3+\frac{1}{2}(b_1\varphi_1^2-b_2\varphi_2^2)\right]\right\}\sin kt.$$

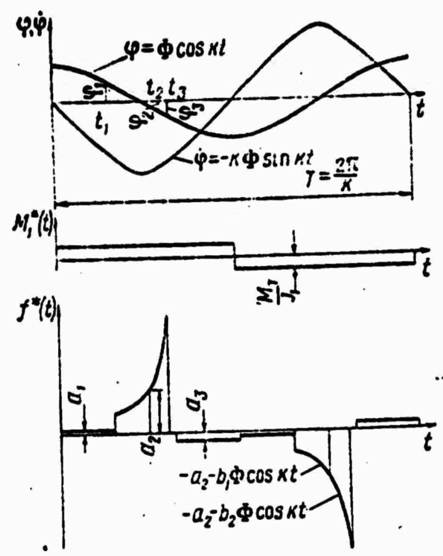

Figure 7
Graphs of the Function

The tooth of the runner impinge on the blades twice during a period. We will denote the impulses of the impinging forces during the time of impact by L_1 and L_2. It is obvious that for the assumption that we have made earlier about the motion of the balance from left to right and and from right to left, these impulses will only differ in their sign (the moments L_1 and L_2 diminish the angular velocity of the balance and due to this fact they are of opposite sign in respect to the respectively angular velocities of the balance.

We have

$$L_1 = \int_{t_3-0}^{t_3+0} M_y(\varphi, \dot{\varphi})\, dt = \int_{-\varphi_3-0}^{-\varphi_3+0} M_y(\varphi, \dot{\varphi})\, \frac{d\varphi}{\dot{\varphi}} = \frac{1}{2} kJ_6(1-s)\sqrt{\Phi^2 - \varphi_3^2}, \qquad (20)$$

where φ_3 —is the angle of rotation of the balance during which the tooth of the runner hits the output blade,

s —is the coefficient of impact, which is equal to the ratio of

the angular velocities of the balance after and before impact,

$k\sqrt{\Phi^2-\varphi_3^2}=|\dot{\varphi}_3|$ — is the angular velocity of the balance before impact which is computed according to the formula (17).

In this manner the function $M_y(\varphi, \dot{\varphi})$ satisfies the conditions

$$\left.\begin{aligned} &M_y(\varphi, \dot{\varphi})=0 \quad \text{при} \quad \varphi=\begin{cases}-\varphi_3\\ \varphi_6,\end{cases}\\ &\int\limits_{-\varphi_3-0}^{-\varphi_3+0} M_y(\varphi, \dot{\varphi})\frac{d\varphi}{\dot{\varphi}}=L_1 \quad \text{при} \quad \varphi=-\varphi_3,\\ &\int\limits_{\varphi_6-0}^{\varphi_6+0} M_y(\varphi, \dot{\varphi})\frac{d\varphi}{\dot{\varphi}}=L_2 \quad \text{при} \quad \varphi=\varphi_6.\end{aligned}\right\} \tag{21}$$

Here φ_6 and $\dot{\varphi}_6$ are the angle of rotation of the balance during which the impact of the tooth of the runner on the input blade takes place, and the angular velocity of the balance before impact, respectively. For the assumption about the character of the motion of the balance that we have made above, we will have

$$\varphi_6=\varphi_3; \quad \dot{\varphi}_6=-\dot{\varphi}_3. \tag{22}$$

We will introduce into consideration the "non-characteristic function" $\delta(\varphi+\varphi_3)$ $-\delta(\varphi-\varphi_6)$, which satisfies the condition

$$\left.\begin{aligned} &\delta(\varphi+\varphi_3)=0 \quad \text{при} \quad \varphi\neq-\varphi_3,\\ &\delta(\varphi-\varphi_6)=0 \quad \text{при} \quad \varphi\neq\varphi_6,\\ &\int\limits_{-\varphi_3-0}^{-\varphi_3+0}\delta(\varphi+\varphi_3)\,d\varphi=1 \quad \text{при} \quad \varphi=-\varphi_3,\\ &\int\limits_{\varphi_6+0}^{\varphi_6-0}\delta(\varphi-\varphi_6)\,d\varphi=1 \quad \text{при} \quad \varphi=\varphi_6.\end{aligned}\right\} \tag{23}$$

Letting

$$M_y(\varphi, \dot{\varphi}) = L_1 \dot{\varphi}\,[\delta(\varphi + \varphi_3) - \delta(\varphi - \varphi_6)], \qquad (24)$$

we will satisfy the conditions (21).

Taking into consideration (24), (23) and (22) we will obtain the following relationships for the determination of Fourrier coefficients of the function $M_y(\varphi, \dot{\varphi})$

$$\left.\begin{aligned} a_0^* &= 0 \\ a_1^* &= -\frac{2\varphi_3 k^2 J_6 \sqrt{\Phi^2 - \varphi_3^2}}{\pi\Phi}(1 - s), \\ b_1^* &= \frac{2k^2 J_6 (\Phi^2 - \varphi^2)}{\pi\Phi}(1 - s). \end{aligned}\right\} \qquad (25)$$

It will now not be difficult to write the Fourrier series for the function M_3^* (t). Actually, on the basis of relationships (18), (15) and (25) we will have

$$\begin{aligned} M_3^*(t) = &-\frac{2\varphi_3 k^2 J_6 \sqrt{\Phi^2 - \varphi_3^2}}{\pi J_1 \Phi}(1 - s)\cos kt + \\ &+ \frac{2k^2 J_6 (\Phi^2 - \varphi_3^2)}{\pi J_1 \Phi}(1 - s)\sin kt. \end{aligned} \qquad (26)$$

By substituting the series we have found into equation (19) and having grouped the coefficients of the sines and the cosines separately in the right side of the equation, we will get

$$\ddot{\varphi} + \omega^2\varphi - \mu_1\varphi^3 = (d_1 + k^2 d_2)\cos kt + (d_3 + k^2 d_4)\sin kt, \qquad (27)$$

$$\left.\begin{aligned} d_1 = &-\frac{M_x}{\pi J_1}\Big\{[2(a_1 - a_2) + b_1\varphi_1]\sqrt{1 - \frac{\varphi_1^2}{\Phi^2}} + (2a_2 + \\ &+ b_2\varphi_2)\sqrt{1 - \frac{\varphi_2^2}{\Phi^2}} + 2a_3\sqrt{1 - \frac{\varphi_3^2}{\Phi^2}} + \Phi\Big[\frac{1}{2}\pi(b_2 - b_1) + \\ &+ b_1 \arccos\frac{\varphi_1}{\Phi} - b_2 \arccos\Big(-\frac{\varphi_2}{\Phi}\Big)\Big]; \end{aligned}\right| \qquad (28)$$

$$
\left.\begin{aligned}
d_2 &= -\frac{2\varphi_3 J_6 \sqrt{\Phi^2-\varphi_3^2}}{\pi J_1 \Phi}(1-s);\\
d_3 &= \frac{4M_T}{\pi J_1} - \frac{2M_x}{\pi J_1}\Big\{a_1 - a_3 - \frac{1}{\Phi}\Big[\varphi_1(a_1-a_2) -\\
&\quad - a_2\varphi_2 - a_3\varphi_3 + \frac{1}{2}(b_1\varphi_1^2 - b_2\varphi_2^2)\Big]\Big\};\\
d_4 &= \frac{2J_6(\Phi^2-\varphi_3^2)}{\pi J_1 \Phi}(1-s).
\end{aligned}\right\} \tag{28}
$$

We will show that the equation (27) can be reduced to the well known Duffing equation /2/: $\ddot{x} + \alpha x + \gamma x^3 = F\cos nt.$

Introducing a new variable

$$\tau = kt + \nu,$$

we will obtain the equation

$$\frac{d^2\varphi}{d\tau^2} + \omega_1^2\varphi - \mu_2\varphi^3 = L_1\cos\tau, \tag{29}$$

where

$$\omega_1^2 = \frac{\omega^2}{k^2}; \quad \mu_2 = \frac{\mu_1}{k^2}; \quad L_1 = \frac{\sqrt{(d_1+k^2d_2)^2 + (d_3+k^2d_4)^2}}{k^2}. \tag{30}$$

The equation of the amplitudinal curves for the Duffing equation has the form

$$n^2 = \alpha + \frac{3}{4}\gamma A^2 - \frac{F}{A},$$

where A is the amplitude of the periodic solution.

For the equation (29) an analogous equation has the form

$$(\omega^2-k^2)A - \frac{3}{4}\mu_1 A^3 = \sqrt{(d_1+k^2d_2)^2 + (d_3+k^2d_4)^2},$$

where $A=\pm\Phi$ (Φ is the amplitude of the vibrations of the balance).

Solving that last equation for the frequency of the balance's vibrations, we will get

$$k=\sqrt{\frac{-\rho_2\overset{+}{(-)}\sqrt{\rho_2^2-\rho_1\rho_3}}{\rho_1}}, \tag{31}$$

where

$$\left.\begin{aligned}\rho_1&=\Phi^2-d_2^2-d_4^2,\\ \rho_2&=-\omega^2\Phi^2+\frac{3}{4}\mu_1\Phi^4-d_1d_2-d_3d_4,\\ \rho_3&=\omega^4\Phi^2-\frac{3}{2}\mu_1\Phi^4\omega^2+\frac{9}{16}\mu_1^2\Phi^6-d_1^2-d_3^2.\end{aligned}\right\} \tag{32}$$

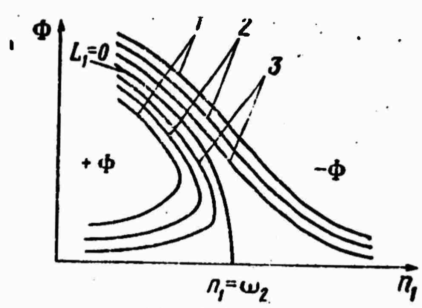

Figure 8
Amplitudinal curves $\Phi=f(n_1)$.

Figure 8 represents some examples of amplitudinal curves (1, 2, 3) for various values of the coefficients d_1, d_2, d_3, d_4. As we can see, for each value of amplitude we have two values of frequency of vibration ff the balance one of which is smaller and the other larger then the corresponding frequency of free vibration ($L_1=0$). However, as was experimentally shown, for the given regulator parameters we witness the establishment of a fully defines frequenccy of vibrations. Since the balance vibrates in a manner very similar to sinusoidal vibrations, the signs of the coefficients ρ_1, ρ_2 and ρ_3, will obviously be determined by the signs of the first terms in the right sides of

the expressions (32) that is $p_1 > 0,\ p_2 < 0,\ p_3 > 0$. Due to this fact, in order to obtain larger values for the frequency of the balance's vibrations, it is necessary to retain the plus sign only in formula (31).

Now it is not difficult to determine tha period of the balance's vibration from the well known formula.

$$T = \frac{2\pi}{k}. \tag{33}$$

Expressions (33), (32), (31) and (28) show that the period of the balance's vibrations is fully and unambigously determined by the parameters of the regulator, and is also dependent upon the amplitude of the balance's vibration (the balance's vibrations are not synchoronous).

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ON THE INFLUENCE OF CLEARANCES IN BEARINGS AND OF THE ECCENTRICITY OF MASSES UPON THE VIBRATIONS OF ROTORS

By

M. F. Khromeyenkov

In the paper of A. K. D'yachkov /2/, M. V. Korovchinskiy /3/ and in other articles (for instance /4/), which are devoted to the computations of bearings with liquid friction it is demonstrated that when the direction of the major vector of the forces that act on the shaft is changing, the centers of the journals are displaced along curvilinear trajectories.

In the computations of the rotors necessary at high speeds, it is ideally assumed that the centers of the journals are stationary, and the size of the initial eccentricities of the rotating masses is also left out of the considerations.

In many cases these procedures actually misrepresents the computed values in comparison with the actual values, due to the fact that the clearances between journals and the bearings influence the position of the former, and serve as a reason for the loading of the shaft by additional centrifugal forces (produced by the curvilinear translational motion in the bearings and in the entire rotor) which change in (value) of the shaft's deflection.

The initial displacement of the center of the rotor's masses relative to the underformed axis of the shaft causes the shaft to deflect even more and

consequently increases the centrifugal force and the gyroscopic moment.

As a result of the action of the above mentioned factors the stability of the rotor's rotation changes. The action of these factors is equivalent to the changing of the rigidity of the shaft. Actually, if the bending moment of the rotating shaft with an equatorial moment of inertia $J(x)$ and with an eccentric mass M is denoted by $M(x, \omega)$ and the bending moment of an ideal shaft (with an ideally balanced mass, stationary journal centers and with the same moment of inertia as the actual shaft) is denoted by $M(x, \omega)$, then the differential equation of the deflection of the shaft can be written in the form

$$y'' = \frac{M_0(x, \omega)}{EJ_*(x)}; \quad J_*(x) = J(x)\frac{M_0(x, \omega)}{M(x, \omega)}, \tag{1}$$

where ω - is the angular velocity of the shaft's rotation,

x - is an ordinate (along the shaft axis),

$J_*(x)$ - is the presented moment of the inertia.

From the inequality $M(x, \omega) > M_0(x, \omega)$ there follows the inequality $J_*(x) < J(x)$ which is equivalent to the diminishing of the shaft's reigidity for the purposes of deflection.

The present paper is dedicated to the computation of the influence of clearances in bearings and to the influence of the initial eccentricity of masses on the value of the critical velocity of the rotor in such a position of the journals that their centers describe circles having radii ρ_A and ρ_B (see fig. 1 and fig. 3); in this paper we consider sliding and rolling bearings as the supports of the rotor.

The above mentioned radii are variable quantities, which are dependent upon a complex of values of physical paramteres of the system consisting of the shaft and its supports, and consequently the trajectories of the centers of the journals, generally speaking, are not circles. But in certain, practically important cases it is possible to consider ρ_A and ρ_B as constant, which permits us to compute the critical velocities, taking into consideration the centrifugal forces due to the translational motion of the journals in their bearings, and also permits us to establish the character of the dynamic loading of the shaft and to compute the value of the bending moment and the deflection which corresponds to this moment.

Statement of Problem

We examine a shaft of a variable cross-section supported on two rigid supported with bearings of different radii. The shaft is loaded by centrifugal forces due to masses distributed along its length as well as due to concentrated masses and is also loaded by gyroscopic moments. We are assuming that the journal turns in the bearing around its axis with the angular velocity ω, and the axis of the journal rotates around the axis of the bearing with the angular velocity ω_p (fig. 1). For the sake of brevity we will call the first rotation the basic one, and the second rotation the additional one.[1]

Each cylindrical element of the shaft, of unit length and mass m is loaded by centrifugal forces due to the two rotations, the basic one and the additional one. Due to this fact the vector $\bar{q}_o$ of the total centrifugal force of the mass m is a geometric sum of the two vectors $\bar{k}$ and $\bar{f}$, the angle between these, in the

general case, changes with the passing of time, which makes the quantity $\bar{q}_o$ variable:

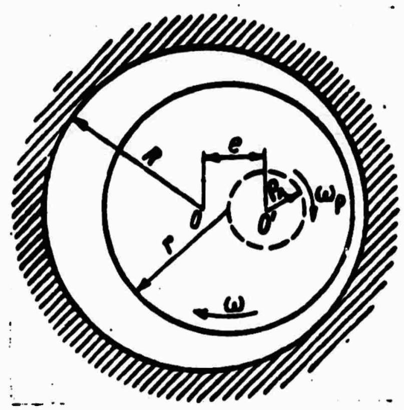

Figure 1
Schematic drawing of the dynamic position of this journal in a bearing with viscous friction.

$$\bar{q}_0 = \bar{k} + \bar{f}, \tag{2}$$

where

$$k = my\nu^2; \quad f = m\rho\dot{\omega}_\rho^2;$$

ν- is the angular velocity of rotation of the plane of deflection around the undeformed axis of the shaft,

y- is the value of the shaft deflection,

ρ- is the radius (simular to ρ_A and ρ_B) corresponding to the mass m.

We will denote the ratio of angular velocities by α and β (β is the coefficient of precession):

$$\alpha = \frac{\omega_\rho}{\omega}; \quad \beta = \frac{\nu}{\omega}. \tag{3}$$

At $|\nu| = |\omega|$ the precession is called synchronized; if the directions of ω and ν are the same, then the precession is called a straight one; if they are different the precession is called a reversed one.

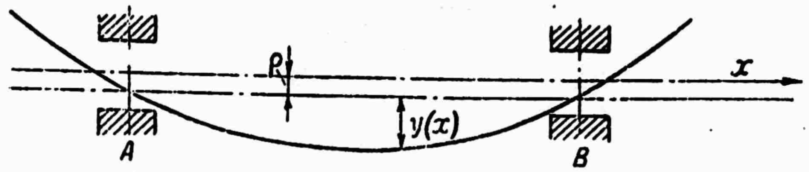

Figure 2
One of the positions of the deflection of a rotating shaft.

The position of the bent shaft is determined by the character of its loading. We will examine two schemes of loading, locating the origin of the coordinates at the left and of the undeformed axis of the shaft.

Let the masses of the shaft and the plates be situated between the supports (fig. 2). In this case the displacement ρ of the undeformed axis of the shaft relative to the line of centers of the circles ρ_A and ρ_B can be considered to be constant only for $\rho_A \neq \rho_B$.

However, if the masses of the shaft and the plates make for a cantilever type of loading (see fig. 3) or $\rho_A \neq \rho_B$ (in the former case), then

$$\rho(0)=\gamma; \quad \frac{\gamma}{a+l_1}=\frac{\rho_A}{l_1}; \quad \gamma=\frac{a+l_1}{l_1}\rho_A;$$

$$\frac{\rho_A}{l_1}=\frac{\rho_B}{l_2}=\delta; \quad \frac{l_1}{l_2}=\frac{\rho_A}{\rho_B}; \quad l_1+l_2=b-a;$$

$$l_1\left(1+\frac{\rho_B}{\rho_A}\right)=b-a; \quad \rho_A=\frac{l_1}{l_1+l_2}(\rho_A+\rho_B);$$

$$l_1=\frac{l_1+l_2}{\rho_A+\rho_B}\rho_B; \quad \delta=\frac{\rho_A+\rho_B}{l_1+l_2};$$

$$\gamma=\frac{a+l_1}{l_1+l_2}(\rho_A+\rho_B); \quad \rho_B=\frac{l_2}{l_1+l_2}(\rho_A+\rho_B)$$

and consequently

$$\rho(x)=\frac{\rho_A+\rho_B}{b-a}(a+l_1-x); \quad \rho_A>0; \quad \rho_B>0. \tag{4}$$

As we can see, in this case ρ is a linear function of x, and the undeformed axis of the shaft describes two coaxial cones, which have a common vertex.

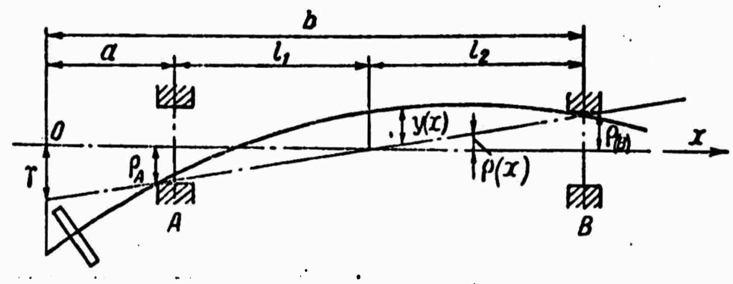

Figure 3
Position of the deflection of a rotating shaft with a mass concentrated on the cantilevered portion.

In all the cases that we have examined the maximum value of the centrifugal force, acting on the mass m is written in the form:

$$q_{max}(x) = m(\alpha^2\rho + \beta^2 y)\,\omega^2.$$

Determination of the Radii ρ_A and ρ_B

Bearings with Liquid Friction

As is well known, when a journal which is turning in the oiled periphery of the bearing is loaded by an external force P whose magnitude and direction are constant, the center of the journal takes a definite position, which is characterized by the value of the bearing's eccentricity which we denote by e.

If, in addition to the force P, another force K whose direction is variable acts on the journal, then the center of the journal performs a

rotation around the point O' (see fig. 1); in this case the trajectory is basically determined by the character of the change of force K.

We will limit ourselves to the investigation of a rotor in which the constant force P is the weight (or another force similar to weight), and the variable force is represented by the centrifugal one, due to the initial eccentricity of the mass and to the deflection of the shaft.

Assuming the oil layer of the bearing is stable /6/ and the above mentioned trajectory sufficiently smooth, we will substitute circular arcs for the elements of the trajectory. The motion of the center of the journal is assumed to be uniform and the angular velocity is taken to be constant. We will in addition examined the condition of balance of the journal on one of the above arcs, assuming that the constant force (the weight) is balanced [2] by a lifting force which is due to the basic rotation of the journal (by the pressure of the oil wedge).

For the sake of definiteness we will examine the bearing A. The mass of the rotor which is acting on the journal, remains eccentric with a certain eccentricity e_r. Due to the fact the motion of this mass in the bearing is a composite one; it consists of the eccentric rotation around the bearing axis, and of the displacement of the axis of the bearing, acting as a weighty section of a straight line, along the circle of radius ρ_A; the first motion gives the centrifugal force K and the second force F_n (fig 4). In addition to the

centrifugal forces K and F_n there acts tn the journal, the force T which is the resistance of the medium (oil) to the journal's advance.

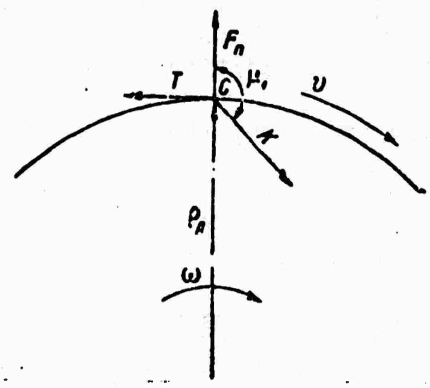

Figure 4
Schematic drawing of forces applied to a journal situated in a bearing.

The active moment which is transmitted by the shaft is balanced by the reactive moment. The angular velocities are in this case considered to be equal (α =1), and the vectore K, F_n and T are rigidly connected (case of a straight synchoronized precession).

From the condition of balance of forces we get:

$$K^2 = F_n^2 + T^2, \qquad (5)$$

where

$K = M_A e_r \omega^2$; $F_n = M_A \rho_A \omega_\rho^2$; M_A — is the mass

we consider the force T to be a function of velocity (see /6/ page 155)

$$T = Ev; \quad v = \omega_\rho \rho_A = \omega \rho_A; \quad E > 0.$$

Substituting the values of the forces into equation (5) and solving the equation for ρ_A, we get

$$\rho_A = \frac{M_A e_r \omega}{\sqrt{(M_A \omega)^2 + E^2}}. \tag{6}$$

It follows from this equality that the dynamical displacement of the journals in bearings with liquid friction is always smaller than the total eccentricity e_r of the rotor. In addition, as can be seen from fig. 4, in the case of bearings with liquid friction, the resultant of the centrifugal forces applied to the rotor is smaller than in the case when the centers of the journals are stationary (bearings without clearances); the damping property of bearings with fluid friction consists of this fact. Formula (6) is the basis for bearing calculations relative to their overloading by the centrifugal forces exerted by the rotor (considering the possibility of direct contact between the working surfaces of the journal and the bearing). The condition of lack of direct contact will be expressed by the inequality.

$$e + \rho_A < R - r, \tag{7}$$

where e is the eccentricity of the bearing and r and R and the radii, respectively, of the journal and the bearing.

We will examine the coefficient E. In order to do this we will takn in the cross-section of the bearing a stationary orthogonal doordinate system ξ, η with its origin at the point O' (see fig. 1) with the axis ξ directed along the eccentricity e. We will write the expressions for the components of the force T:

$T_{\xi\xi} = E_{\xi\xi} \frac{d\xi}{dt}$ — is a resistive force directed along the axis ξ, which arises during the motion of the journal around this axis ξ and which acts in the direction opposite to the direction of motion

(see /6/, page 156),

$T_{\eta\xi} = E_{\eta\xi}\frac{d\eta}{dt}$ — is a resisting force, directed along the axis ξ which arises during the motion of the journal around axis η,

$T_{\xi\eta} = E_{\xi\eta}\frac{d\xi}{dt}$ — is a resisting force which is directed along the axis η which arises during the motion of the journal around axis ξ,

$T_{\eta\eta} = E_{\eta\eta}\frac{d\eta}{dt}$ — is a resistive force, directed along axis η and which arises during the motion of the journal around axis η.

Turning to the projections of the force T on the coordinate axes, we get

$$T_\xi = T_{\xi\xi} + T_{\eta\xi} = E_{\xi\xi}\frac{d\xi}{dt} + E_{\eta\xi}\frac{d\eta}{dt};$$

$$T_\eta = T_{\eta\eta} + T_{\xi\eta} = E_{\eta\eta}\frac{d\eta}{dt} + E_{\xi\eta}\frac{d\xi}{dt}; \quad E_{\xi\eta} = E_{\eta\xi}.$$

Taking into consideration the relationships

$$\xi = \rho_A \cos\varphi; \quad \eta = \rho_A \sin\varphi; \quad \frac{d\varphi}{dt} = \omega, \tag{8}$$

we get

$$\frac{d\xi}{dt} = -\rho_A\omega\sin\varphi; \quad \frac{d\eta}{dt} = \rho_A\omega\cos\varphi;$$

$$T_\xi = \omega\rho_A(E_{\xi\eta}\cos\varphi - E_{\xi\xi}\sin\varphi);$$

$$T_\eta = \omega\rho_A(E_{\eta\eta}\cos\varphi - E_{\eta\xi}\sin\varphi);$$

$$T = \sqrt{T_\xi^2 + T_\eta^2} =$$

$$= \omega\rho_A\sqrt{(E_{\xi\eta}\cos\varphi - E_{\xi\xi}\sin\varphi)^2 + (E_{\eta\eta}\cos\varphi - E_{\eta\xi}\sin\varphi)^2}. \tag{9}$$

Noting that

$$\omega\rho_A = v; \quad T = Ev,$$

we will have

$$E^2 = (E_{\xi\xi}^2 + E_{\eta\xi}^2)\sin^2\varphi + (E_{\eta\eta}^2 + E_{\xi\eta}^2)\cos^2\varphi - -2(E_{\xi\xi} + E_{\eta\eta})E_{\xi\eta}\sin\varphi\cos\varphi. \tag{10}$$

It is obvious that the quantity E^2 and, as follows from the equality (6), the radius ρ_A, change with the rotation of the center of the journal around the

point O' (see fig. 1).

In order to check the loading of the bearing according to the inequality (7) it is necessary and sufficient to know the value of E^2 when the velocity v is directed perpendicular to the axis ξ (to the l line of the eccentricity e), that is, at the instant when the surface of the journal is nearest to the surface of the bearing, in this case $\varphi=0$ and consequently

$$E^2=E^2_{\eta\eta}+E^2_{\eta\xi};\quad E=\sqrt{E^2_{\eta\eta}+E^2_{\eta\xi}}. \tag{11}$$

The components of the coefficient of resistance E which correspond to the above mentioned direction of the velocity v, which were obtained by S. A. Chernavski for bearings having an angle of coverage by the oil layer equal to 180°, are of the following form

$$\left.\begin{aligned} E_{\eta\eta}&=\frac{2\mu l}{\psi^3\chi}k'\sqrt{1-\chi^2};\\ E_{\eta\xi}&=\frac{2\mu l}{\psi^3}k', \end{aligned}\right\} \tag{12}$$

where

l – is the working length of the bearing,

d – is the diameter of the journal,

μ – is the absolute coefficient of viscosity,

$\chi=\frac{l}{R-r}$ – is the relative eccentricity of the bearing,

$\psi=\frac{R-r}{r}$ – is the relative radical clearance,

$k'=\frac{k\psi^2}{\mu\omega}$ – is the diverted load,

$k=\frac{P}{ld}$ – is the specific load,

In the case when $\chi=0$, the resistance coefficient will be equal, according to /6/ to

$$E=\frac{12\pi\mu l}{\psi^3}. \tag{13}$$

As can be seen from the equality (6), in order to determine the quantity P_A it is necessary to know the value of the eccentricity of the rotor's masses, which are transferred to the journal of the bearing A. We will examine as an example a shaft with a disc situated between the supports (fig. 5).

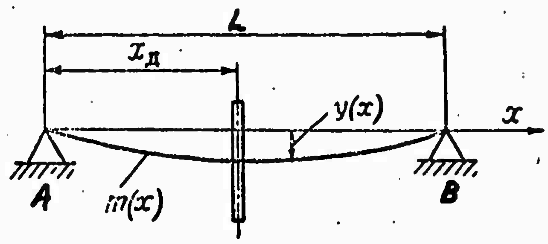

Figure 5
Schematic drawing showing the bringing of the rotor's masses to the journal

The coordinates of the center of gravity of the disc are x_d and $e_d=y_d+e_1$, where e_1 is the initial eccentricity of the mass of the disc.

The coordinates of the center of gravity of the shaft are determined by the integrals

$$x_в=\frac{\int_0^L xm(x)\,dx}{\int_0^L m(x)\,dx}; \quad e_в=\frac{\int_0^L y(x)\,m(x)\,dx}{\int_0^L m(x)\,dx}. \tag{14}$$

The coordinates of the center of gravity of the entire rotor are determined by the equalities

$$x_r=\frac{M_в x_в+M_д x_д}{M_r}; \tag{15}$$

$$e_r=\frac{M_в e_в+M_д e_д}{M_r}, \tag{16}$$

where M_s and M_d are the masses of the shaft and the disc, respectively; $M_r=M_s+M_d$ is the mass of the entire rotor, which distributes itself on the journal, in the same manner as weight is distributed between supports

$$M_A = \frac{L - x_r}{L} M_r; \quad M_B = \frac{x_r}{L} M_r. \tag{17}$$

For M_A and M_B there will exist only one eccentricity e_r.

Roller Bearings

The trajectory of motion of roller bearings has not been studied up to now and it is doubtful whether this problem can have a satisfactory solution, due to the fact that frictional force in this case remains undetermined.

However, it is possible to assume with accuracy sufficient for practical applications, that, during the rotor's rotation, the center of the journal performs one of these motions: it either describes a circle with a radius equal to the radial clearance of the bearing, or it swings like a pendulum in the boundaries of the lower part of the above mentioned circumference. The first type of motion takes place when the dynamic component of a bearing reaction (of the force K) is greater than the static component (of the force P), and the second type takes place when the opposite is true - then the static component is greater than the dynamic one. In both cases the trajectory of the journal's center can be considered to be a circumference with a radius

equal to the radial clearing of the bearing; in this case the center of 0 and 0' coincide (see fig. 1).

$$\rho_A = R - r = \text{const.} \tag{18}$$

Calculation of the Influence of the Initial Eccentricity of the Rotor's Masses On the Critical Speeds

Some of the eccentric parts of the rotor (disc, muffs, etc) are sometimes seated on short sections of the shaft; due to this fact we assign their masses to the separate sections of the shaft and we compute the centrifugal forces according to the formula

$$Q_i = M_i \omega^2 [\alpha^2 \rho_i + \beta^2 (e_i + y_i)]; \quad i = 1, 2, 3, \ldots p, \tag{19}$$

where M_i — is the mass,

y_i —is the deflection of the shaft,

e_i —is the eccentricity,

ρ_i —is the displacement of the undeformed axis

ω —is the angular velocity.

The other parts are fastened to relatively long portions of the shaft and due to this we consider their masses to be distributed along the shaft's length.

It is obvious that the additional centrifugal forces, which arise as a result of the eccentricity of the rotating masses can be considered as external periodical loads with a frequency equal to the angular velocity of the basic rotation of the shaft. The latter permits us to compute the influence

of the centrifugal forces upon the stability of motion by the quantities of masses (?).

In order to clarify the physical picture, we will first examine a rotor having one mass M_1 and an eccentricity e_1; and the value of the displacement of the undeformed shaft axis at the section underneath this mass is ρ_1. We consider the displacement ρ_1 to be situated in the plane of the shaft's deflection. The centrifugal force will in this case be written in the form

$$Q_1 = M_1\omega^2[\alpha^2\rho_1 + \beta^2(e_1 + y_1)]. \tag{20}$$

The effect of the initial eccentricity of the mass M_1 will be expressed by the force ΔQ_1:

$$\Delta Q_1 = Q_1 - M_1\omega^2(\alpha^2\rho_1 + \beta^2 y_1) = M_1\omega^2\beta^2 e_1. \tag{21}$$

Upon dividing ΔQ_1 by the radial acceleration j of an non-eccentric mass, we will obtain the apparent additional mass

$$\frac{\Delta Q_1}{j} = \frac{\omega^2\beta^2 e_1 M_1}{\omega^2(\alpha^2\rho_1 + \beta^2 y_1)} = \frac{\beta^2 e_1 M_1}{\alpha^2\rho_1 + \beta^2 y_1}, \tag{22}$$

by whose magnitude we increase the numerical magnitude of the mass M_1 on applying it to the line of deflection of the shaft.

Consequently, the magnitude of the applied mass will be

$$M_{(1)} = M_1\left(1 + \frac{\beta^2 e_1}{\alpha^2\rho_1 + \beta y_1}\right). \tag{23}$$

In the same manner it is possible to bring to the line of deflection any number of masses, whose eccentricities lie in one plane, and to introduce the above mentioned masses into the computation.

(The actual value of the mass should enter into the computations of the gyroscopic moment).

The relative positions of the values e_i is determined only in the process of the construction of the rotor, and only the absolute values of e_i can be assigned in the design stage. Due to this fact, all the e_i should be considered to be parallel (in one plane and on one side of the shaft's axis) for the design purposes.

In certain cases, when the eccentricities e_i are not parallel and the products $M_i e_i$ are comparatively large, it is possible to bring the masses to the line of deflection by an approximate method, which is presented below and which is useful only for the computation of lower critical speed.

Let, for instance, the rotor contain two masses M_1 nad M_2 with eccentricities e_1 and e_2, making an angle θ with one another, and situated at different cross-sections of the shaft. With the same accuracy that it is possible to consider the surface containing the line of deflection of the shaft to be a plane, with the same accuracy it is possible to determine this plane in the following manner.

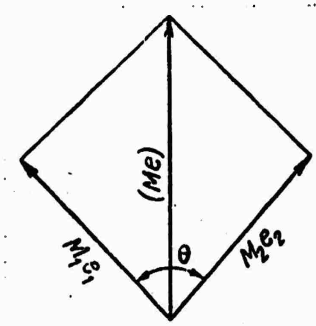

Figure 6
Schematic drawing showing the bringing of eccentric masses of the rotor to the axis of rotation.

Considering e_1 and e_2 to be vectors, we project the products M_1e_1 and M_2e_2 on a plane perpendicular to the undeformed axis of the shaft (fig. 6). Having denoted the sum of the above vectors by (Me), we have

$$(Me) = \sqrt{(M_1e_1)^2 + (M_2e_2)^2 + 2(M_1e_1)(M_2e_2)\cos\theta}. \tag{24}$$

We will divide the quantity (Me) into two parts, proportional to M_1e_1 and M_2e_2

$$(Me)_1 = \frac{M_1e_1}{M_1e_1 + M_2e_2}(Me); \tag{25}$$

$$(Me)_2 = \frac{M_2e_2}{M_1e_1 + M_2e_2}(Me). \tag{26}$$

In this manner, the products of the masses and their eccentricities are brought to a single plane, formed by the undeformed axis of the shaft and the vector (Me). The masses that have been brought to the plane in this case will be

$$\left.\begin{aligned} M_{(1)} &= \frac{(Me)_1}{e_1}\left(1 + \frac{\beta^2 e_1}{\alpha^2\rho + \beta^2 y_1}\right) \quad e_1 > 0; \\ M_{(2)} &= \frac{(Me)_2}{e_2}\left(1 + \frac{\beta^2 e_2}{\alpha^2\rho_2 + \beta^2 y_2}\right) \quad e_2 > 0. \end{aligned}\right\} \tag{27}$$

By the method of a resultant vector it is possible to bring to the shaft's axis any number of masses both concentrated and distributed over short sections.

We will not compute the dynamical unbalance. We will imagine a disc with mass M_1 whose plane is inclined to the axis of the mass' rotation by the angle $\theta < \frac{\pi}{2}$. It is obvious that in this case the centrifugal forces of the top and bottom halfs of the disc will result in a couple (moment):

$$M_{g1} = 2\lambda^2 \sin\vartheta \cos\vartheta \int_0^R r^2\, dM_1 = \lambda^2 J_s \sin\vartheta \cos\vartheta, \qquad (28)$$

where

$$\lambda = (1-\beta)\,\omega.$$

In practice the angle ϑ has a value close to $\frac{\pi}{2}$ which permits us to assume

$$\sin\vartheta \simeq 1; \quad \cos\vartheta \simeq \frac{\pi}{2} - \vartheta = \eta$$

and, consequently

$$M_{g1} = \lambda^2 J_s \eta = \omega^2 (1-\beta)^2 J_s \eta; \quad \eta \gtrless 0, \qquad (29)$$

where J_s - is the equatorial (in respect to the diameter) moment of inertia,

ω - is the angluar velocity of the disc,

β - is the coefficient of precession from (3).

Assuming that the angle η and the angle of inclination of the disc are loacted in the same plane as a result of the shaft's deflection, we sum M_{g1} together with the gyroscopic moment of the mass M_1.

Determination of the Critical Speeds of the Rotor

We will investigate the general case of precessional motion. Let there take place in addition to the rotation of the undeformed axis with the velocity ω_φ, the roation of the plane of the shaft's deflection with the angular velcoity ν, and its cross-ection should rotate relative to this plane with the angular velocity λ. In other words,

let the basic velocity of the shaft be broken up into two component vectors

$$\bar{\omega} = \bar{\nu} + \bar{\lambda}. \tag{30}$$

We will present the angular velocities ν and λ in the form (see /1/)

$$\nu = \beta\omega; \quad \lambda = (1 - \beta)\omega, \tag{31}$$

where β is the coefficient of procession from (3).

Each shaft element is loaded by centrifugal forces die to the basic and additional rotations.

As has been shown above, in the general case ($\alpha \neq \beta$), the angle between the vectors of the centrifugal forces $\bar{k}$ and $\bar{f}$ changes with the passage of time. The moduli of k and f periodically form an arithmetic sum and a difference, as a result of which the shaft acquires cross-sectional pulsational vibrations.

It is easy to see (fig. 7) that

$$q_0 = \sqrt{k^2 + f^2 + 2kf\cos\mu_2}, \tag{32}$$

where

$$\mu_2 = (\omega_\rho - \beta\omega)t = (\alpha - \beta)\omega t;$$

and t is time.

It is known that the critical shaft speed corresponds to its maxiuum deflection, which obviously corresponds to the maximum value of the vector $\bar{q}_0$, when the moduli k and f represent an arithmetical

sum.

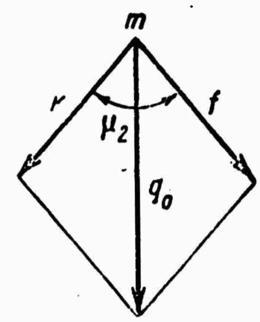

Figure 7
Schematic drawing showing the loading of a unit length of the shaft by centrifugal forces.

By fixing the instant of time of the summation of k and f, we can superpose the plane of the shaft's deflection on the vertical plane, directing the vector $\bar{q}_{0max}$ downward, and we will be able to find the forces which load an element of the shaft having a mass m(x):

centrifugal force due to the motion of the undeformed axis of the shaft round the axis of the bearings

$$\alpha^2 m(x)\,\omega^2 \rho(x);$$

centrifugal force due to the precessional motion

$$\beta^2 m(x)\,\omega^2 y(x),$$

where y(x) is the shaft's deflection, force of gravity

$$gm(x) \quad (g=981 \text{ см/сек}^2);$$

inertial force due to the cross-sectional vibrations (disturbances due to the weight and the pulsations)

$$-m(x)\frac{d^2v}{dt^2}; \quad \frac{d^2v}{dt^2}<0,$$

where v(x) is the amplitude of the bending vibrations (that is the dis-

The total load on m(x) is written in the form

$$q(x)=\omega^2 m(x)\left[\alpha^2\rho(x)+\beta^2 y(x)+\frac{g}{\omega^2}-\frac{1}{\omega^2}\frac{d^2v}{dt^2}\right] \tag{33}$$

or

$$q(x)=\omega^2 m(x)u(x), \tag{34}$$

where

$$u(x)=\alpha^2\rho(x)+\beta^2 y(x)+\frac{g}{\omega^2}-\frac{1}{\omega^2}\frac{d^2v}{dt^2}. \tag{35}$$

We will call the function u(x) the generalized deflection of the shaft.

Disregarding the last component in (35), we have,

$$u(x)\simeq\alpha^2\rho(x)+\beta^2 y(x)+\frac{g}{\omega^2}, \tag{36}$$

from where

$$u''(x)=\beta^2 y''(x),$$

since

$$\rho''(x)\equiv 0$$

and, consequently

$$y''(x)=\frac{u''(x)}{\beta^2}. \tag{37}$$

In addition to the centrifugal forces due to the distributed mass m(x), there act on the shaft centrifugal forces due to the concentrated masses and their gyroscopic moments (we do not consider the gyroscopic moments of the distributed masses):

$$\left.\begin{aligned}&Q_i(c_i)=\omega^2 M_i u(c_i);\\&M_i^{(r)}(c_i)=\omega^2 J_{s,i}\xi_i;\quad i=1,2,3,\ldots,p,\end{aligned}\right\} \tag{38}$$

placement from the deflection curve).

where M_i — is the mass of the disc

J_{3i} — is the equatorial moment of inertia

c_i — is an ordinate (see fign. 8)

$$\xi_i = \beta(2-\beta)y'(c_i) + \rho'(c_i) + (1-\beta)^2\eta. \tag{39}$$

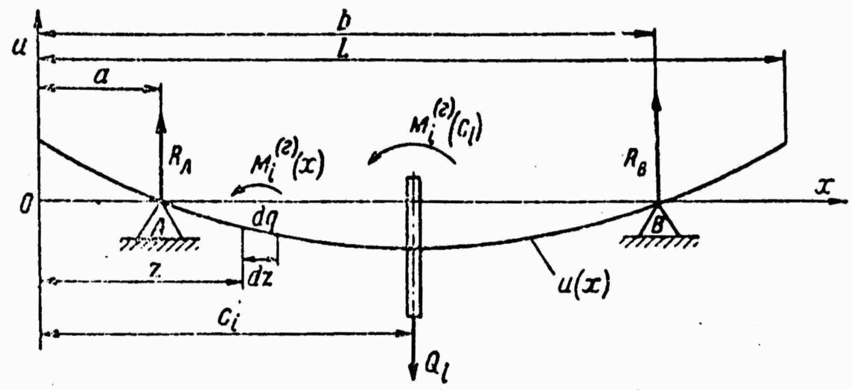

Figure 8
Schematic drawing showing the dynamic loading of the shaft.

The bending moment acting on the shaft due to the distributed load will be written in the form given in /5/

$$M_g = \omega^2 \int_0^x m(z)\,u(z)\,(x-z)\,dz. \tag{40}$$

In the given problem we assume that the moments of inertia of the distributed masses are relatively small. This permits us not to take into consideration their gyroscopic moments $M^{(g)}(x)$, which significantly simplifies the problem. The computation of the moments $M^{(g)}(x)$ reduces the problem of determination of the critical speeds to a system of two integro-differnntial equations as shown in /1/.

The total bending moment in the cross-section having the coordinate s is written in the form

$$M(x)=\omega^2\int_0^x m(z)\,u(z)(x-z)\,dz+ +S(a)R_A(x-a)+S(b)R_B(x-b)+ +\sum_1^p S(c_i)\,Q_i(x-c_i)+\sum_1^p M_i^{(r)}(c_i), \tag{41}$$

where R_A and R_B are the reaction of the supports A and B.

$S(a)$, $S(b)$, $S(c_i)$ -- are unitary functions /1/, and also

$$\left.\begin{aligned} &S(a)=\begin{cases}0 & \text{при } x\leqslant a;\\ 1 & \text{при } x>a;\end{cases} \qquad S(b)=\begin{cases}0 & \text{при } x\leqslant b\\ 1 & \text{при } x>b;\end{cases}\\ &S(c_i)=\begin{cases}0 & \text{при } x\leqslant c_i;\\ 1 & \text{при } x>c_i;\end{cases} \qquad i=1,2,3,\ldots p.\end{aligned}\right\} \tag{42}$$

The reactions of the supports are obtained from the condition of equilibrium of the system of forces around the right end of the shaft ($x=L$):

$$\omega^2\int_0^L m(z)\,u(z)\,dz+R_A+R_B+\sum_1^p Q_i=0; \tag{43}$$

$$\omega^2\int_0^L m(z)\,u(z)(L-z)\,dz+R_A(L-a)+R_B(L-b)+ +\sum_1^p Q_i(L-c_i)+\sum_1^p M_i^{(r)}(c_i)=0. \tag{44}$$

From the equations (43) and (44) we get

$$R_A=\frac{\omega^2}{b-a}\left[\int_0^L m(z)\,u(z)\,z\,dz-b\int_0^L m(z)\,u(z)\,dz\right]- -\sum_1^p Q_i\frac{b-c_i}{b-a}-\sum_1^p\frac{M^{(r)}(c_i)}{b-a}; \tag{45}$$

$$R_B = \frac{\omega^2}{b-a}\left[a\int_0^L m(z)\,u(z)\,dz - \int_0^L m(z)\,u(z)\,z\,dz\right] + \tag{46}$$

$$+\sum_1^p Q_i\frac{a-c_i}{b-a} + \sum_1^p \frac{M_i^{(r)}(c_i)}{b-a}.$$

In order to determine the quantity ω^2 we will set up a marginal (limiting?) integral equation. Without taking into consideration the transverse (shearing?) force, the differential equation of the shaft's deflection has the form

$$u''(x) = \omega^2\varphi(x), \tag{47}$$

where $\varphi(x) = \frac{\beta^2 M(x)}{\omega^2 EJ(x)}, (\beta \neq 0),$ and in actuality is independent of ω^2; due to this fact the bending moment M(x) is a linear function of ω^2.

Integrating (47), we get

$$u'(x) = \omega^2\int_0^x \varphi(z)\,dz + u'(0); \tag{48}$$

$$u(x) = \omega^2\int_0^x\left[\int_0^x \varphi(z)\,dz\right]dx + u'(0)\,x + u(0). \tag{49}$$

The last equation is a marginal homogeneous integral equation /1/, and ω^2 is it characteristic number. The equation is called homogeneous if it satisfies u=0; in the given case this condition is fulfilled if

$$\eta = \frac{\pi}{2} - \vartheta \approx 0.$$

If $\eta \gg 0$ or $\eta \ll 0$, then ϑ it is necessary to integrate a non-homogeneous marginal equation; the method of integration is presented in the monograph /1/.

We are examining a case when the bearing clearances are of various widths and the following conditions are imposed on the function u(x)

$$u(a)=\alpha^2\rho_A+\frac{g}{\omega^2};\quad u(b)=\alpha^2\rho_B+\frac{g}{\omega^2}. \tag{50}$$

On the basis of the last equalities we get

$$u(0)=\frac{\omega^2}{b-a}\left\{a\int_0^b\left[\int_0^x\varphi(z)\,dz\right]dx-b\int_0^a\left[\int_0^x\varphi(z)\,dz\right]dx\right\}+ \\ +\frac{\alpha^2(b\rho_A-a\rho_B)}{b-a}+\frac{g}{\omega^2}; \tag{51}$$

$$u'(0)=\frac{\omega^2}{b-a}\left\{\int_0^a\left[\int_0^x\varphi(z)\,dz\right]dx-\int_0^b\left[\int_0^x\varphi(z)\,dz\right]dx\right\}+ \\ +\frac{\alpha^2(\rho_B-\rho_A)}{b-a}. \tag{52}$$

In this manner this function u(x) can be presented in the form

$$u(x)=\omega^2\int_0^x\left[\int_0^x\varphi(z)\,dz\right]dx+\frac{\omega^2}{b-a}\left\{\int_0^a\left[\int_0^x\varphi(z)\,dz\right]dx- \right. \\ \left. -\int_0^b\left[\int_0^x\varphi(z)\,dz\right]dx\right\}x+\frac{\omega^2}{b-a}\left\{a\int_0^b\left[\int_0^x\varphi(z)\,dz\right]dx- \right. \\ \left. -b\int_0^a\left[\int_0^x\varphi(z)\,dz\right]dx\right\}+\frac{\alpha^2(\rho_B-\rho_A)}{b-a}x+\frac{\alpha^2(b\rho_A-a\rho_B)}{b-a}+\frac{g}{\omega^2}. \tag{53}$$

From equation (36) we have

$$y(x)=\frac{u(x)}{\beta^2}-\left(\frac{a}{\beta}\right)^2\rho(x)-\frac{g}{\beta^2\omega^2}. \tag{54}$$

In this manner the dynamic problem of determining the critical speed of the rotor reduces to the problem of the static deflection of the shaft loaded by a system of forces and moments. Reducing the problem of the deflection to an integral equation, we will find the function u(x) and consequently y(x) and ω^2 by the method of successive approximations presented in I. A. Birger's monograph /1/. In this case we find the value of the characteristic number ω^2_K from the condition of the minimum quadratic deflection:

$$\varepsilon=\int_0^L [u_{(j)}(x)-u_{(j-1)}(x)]^2 h_0(x)\,dx;$$
$$\frac{\partial\varepsilon}{\partial\lambda}=0;\quad \lambda=\omega_k^2, \tag{55}$$

where $u_{(j-1)}(x)$, $u_{(j)}(x)$ — are two consecutive approximations of the function u(x),

$h_0(x)$ — is the "weight" of the quadratic deflection (we can practically assume that $h_0(x)=1$.

We will introduce denotations

$$\int_0^x\left[\int_0^x \varphi(z)\,dz\right]dx+\frac{x}{b-a}\left\{\int_0^a[\varphi(z)\,dz]\,dx-\right. \tag{56}$$

$$-\int_0^b\left[\int_0^x \varphi(z)\,dz\right]dx+\frac{1}{b-a}\left\{a\int_0^b\left[\int_0^x \varphi(z)\,dz\right]dx-\right.$$

$$\left.-b\int_0^a\left[\int_0^x \varphi(z)\,dz\right]dx\right\}=\Phi(x); \tag{57}$$

$$\frac{a^2(\rho_B-\rho_A)}{b-a}x+\frac{a^2(b\rho_A-a\rho_B)}{b-a}+\frac{g}{\omega^2}=F(x);$$
$$u_{(j-1)}(x)-F(x)=f(x); \tag{58}$$

$$f(x)\,\Phi(x)\,h_0(x)=P(x);\quad \Phi^2(x)\,h_0(x)=Q(x). \tag{59}$$

Differentiating $\mathcal{E}$ in respect to χ and equating the derivatives to zero we will get

$$\int_0^L [u_j(x)-u_{(j-1)}(x)]\frac{\partial u_j}{\partial \chi}h_0(x)\,dx=0; \tag{60}$$

$$\int_0^L [\chi\Phi(x)-u_{(j-1)}(x)+F(x)]\,\Phi(x)\,h_0(x)\,dx=0. \tag{61}$$

We open the brackets in the last equation

$$\chi\int_0^L \Phi^2(x)\,h_0(x)\,dx=\int_0^L [u_{(j-1)}(x)-F(x)]\,\Phi(x)\,h_0(x)\,dx, \tag{62}$$

from where

$$\chi\int_0^L Q(x)\,dx=\int_0^L P(x)\,dx;\quad \chi=\omega_k^2$$

or

$$\chi=\omega_k^2=\frac{\int_0^L P(x)\,dx}{\int_0^L Q(x)\,dx}. \tag{63}$$

In practice the integration can always be performed numerically.

As an initial function u(x) we take a parabola, and we compute the value of the component Φ/ω^2 of function F (x), starting with the first approximation $u_1(x)$:

$$u_0(x)=Ax^2+Bx+C, \tag{64}$$

A, B, and C are determined from the conditions

$$
\left.\begin{aligned}
u_0(a) &= Aa^2 + Ba + C = \alpha^2 \rho_A; \\
u_0(b) &= Ab^2 + Bb + C = \alpha^2 \rho_B; \\
u_0(x_*) &= Ax_*^2 + Bx_* + C = \sigma(\rho_A + \rho_B).
\end{aligned}\right\} \qquad (65)
$$

where x_* —is the coordinate of a section corresponding to the maximum deflection of the shaft,

σ —is a coefficient, equal to, or greater then, unity (as the ratio of the mean value of the moment of inertia to the length of the shaft decreases, the value of σ increases).

Puttign $u_0(x)$ underneath the integral signs in the equation (65), we obtain $u_1(x)$ as a solution of equation (53) we get $u_2(x)$ etc, in similar manner. As is pointed out by I. A. Birger, the process of approximations reduces sufficiently fast to the solution of equation (53). Once we have u(x) we can compute the centrifugal forces and the moments which act on the shaft.

We will now determine the second critical speed. Let the integral equation (53) be solved; that is, let the first characteristic number $\chi^{(1)} = \omega_K^2$ and the first characteristic function $u^{(1)}(x)$ be determined. We will denote the process of passing from the approximation with the index n to the approximation with the index n+1 as the operator K1, then

$$u_{n+1}^{(1)} = \chi_{n+1}^{(1)} K_1 u_n^{(1)}; \quad n = 1, 2, 3, \ldots \qquad (66)$$

We will find the second characteristic number $\chi^{(2)}$ and the second characteristic function $u^{(2)}(x)$ from the equation

$$u_{n+1}^{(2)}(x) = \chi_{n+1}^{(2)} K_2 u_n^{(2)}(x), \qquad (67)$$

where

$$K_2 u_n^{(2)}(x) = K_1 u_n^{(2)}(x) - u^{(1)}(x) \frac{\int_0^L K_1 u_n^{(2)}(x) u^{(1)}(x) h(x)\,dx}{\int_0^L [u^{(1)}(x)]^2 h(x)\,dx}.$$

In this case it is more convenient to use for the initial function $u_0^{(2)}(x)$ a sinusoid, which has a zero value at $x = \frac{1}{2}(a+b)$ and which satisfies the initial conditions, taking into consideration the position of the journals in the bearings (one journal below the axis of the bearings and the other above).

In the determination of the critical speed of order j we solve the integral equation

$$u_{n+1}^{(j)}(x) = \chi_{n+1}^{(j)} K_j u_n^{(j)}(x), \qquad (68)$$

where

$$K_j u_n^{(j)}(x) = K_1 u_n^{(j)}(x) - \sum_{i=1}^{j-1} u^{(i)}(x) \frac{\int_0^L K_1 u_n^{(j)}(x) u^{(i)}(x) h(x)\,dx}{\int_0^L [u^{(i)}(x)]^2 h(x)\,dx}.$$

Conclusions

1. We have developed a method for the determination of critical speeds of an eccentrical rotor in the case of radial displacement of the shaft's journals in their bearings.

2. The examination by approximate methods of the precessional motion of the journal in a bearing with liquid friction, shows that the dynamic displacement of the journal's shaft center is always smaller than the total eccentricity of the rotor's masses. The resultant of the centrifugal forces acting on the rotor is in this case smaller than in the case of ideal supports (when the centers of the journals are stationary). As a result of the diminution of the centrifugal forces, the shaft acts as if it had become stiffer, as far as deflection is concerned. Consequently, in the sense of the critical speeds of the rotor the displacement of journals in bearings with liquid friction should be considered as a positive factor (for the case of a stable oil layer).

3. In the case of displacement of the journal center in a roller bearing along the arc of the circle ρ =R-r (r is the journal radius, R. is the bearing radius) the rotor is loaded by an additional centrifugal force which adds to the centrifugal force due to the basic rotation (around the undeformed axis of the shaft), and their resultant is always greater than in the case of ideal supports (with stationary journal centers).

As a result the shaft acts as it wewe becoming more susceptible to deflection and consequently its critical speed is lowered. In this case the additional rotation of the shaft's sections changes the gyroscopic moments of the rotating masses of the rotor (the same result is obtained from the bending of the undeformed axis of the shaft in the bearings) which in turn influences the value of the critical angular velocity.

The author feels obligated to express his deeply appreciation to professor V. V. Dobronravov, who has lent an extremely helping hand in the preparation of this article.

FOOTNOTES

1. Page 249. Such a rotation can arise due to the external frictional forces acting on the rotor, the non-equilibrium of centrifugal or aerodynamic forces exerted by the discs, etc. As an example of such a rotation we can consider a straight synchronous precession, in which case $\omega_p = \omega$.

2. Page 253. This assumption is permissible due to the relatively small deflection of the journal from its position, in the case of static loading.

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ON THE STABILITY OF AN AIPLANE FOR THE CASE OF MOTION OF ITS FUEL IN THE TANKS

By

K. A. Golenko

We will formulate the following linear statement of the problem.

An airplane flies along a portion of its trajectory almost horizontally During non-turbulent motion the axis of symmetry of the airplane Ox is forward-directed and coincides with the vector of the center of gravity of the body. Axis Oz is directed downward, and the axis Oy along the right wing.

The velocity of the center of gravity in turbulent flight is presented in the form

$$\overline{V}=\overline{V}_0+\overline{u}, \tag{1}$$

where $\overline{V}_0$ is the velocity of the center of gravity of the airplane before the disturbance and u is the disturbance, a quantity of the first order of smallness.

The maximum angular velocity of the airplane $\overline{\omega}(\dot{\varphi}, \dot{\theta}, \dot{\psi})$, in its projection on the axes x,y and z is a quantity of the first order of smallness. It is necessary to set up the equations of the disturbance system consisting of the airplane and the liquid, and to reduce them to a system of ordinary differential equations.

The equations of the relative motion of a liquid in any given tank (x_1, x_2, y_1, y_2 are the coordinates of the (tank) walls, z_1 is the coordinate of the (tank) bottom and z_2 is the undisturbed free surface) have the form

$$\frac{\partial \bar{v}_1}{\partial t} + \frac{d\bar{V}}{dt} + \frac{d\bar{\omega}}{dt} \times \bar{r} = \bar{F} + \frac{1}{\rho} \operatorname{grad} p; \tag{2}$$

$$\operatorname{div} \bar{v} = 0, \tag{3}$$

where $\bar{v}_1$ is the vector of the relative velocity of the liquid, whose value is of the first order of smallness, and p is the pressure.

Values of the second order of smallness were disregarded in equation (2).

We will take the equation of the free surface in the form

$$z_2 - z = \varkappa_1 (x, y, t), \tag{4}$$

where $\varkappa$ is a quantity of the first order of smallness.

At the free surface we have the conditions

$$p = p_0; \tag{5}$$

$$\frac{\partial \varkappa_1}{\partial t} = -v_{1z}. \tag{6}$$

We will note that the major vector of the quantity of motion of the system consisting of the airplane and the liquid consists of the main vector of the quantity of motion of the body of the plane of the fluid in translational motion and of the main velocity of the quantity of motion of the fluid in its

relative motion. It is possible to present the main moment of the quantity of motion in an analogous manner.

We will take the above mentioned coordinate system Oxyz, which is rigidly tied to the body of the plane, then the equations of motion of the system, disregarding terms of the second order of smallness can be written in the form

$$\begin{aligned} &\frac{d\overline{Q}}{dt}+\overline{\omega}\times\overline{Q}+\sum_{j=1}^{s}\frac{d}{dt}\int_{\tau}\rho_j\overline{v}_j\,d\tau=\overline{F};\\ &\frac{d\overline{K}}{dt}+\overline{\omega}\times\overline{K}+\sum_{j=1}^{s}\frac{d}{dt}\int_{\tau}\rho_j(\overline{r}\times\overline{v}_j)\,d\tau=\overline{M}, \end{aligned} \tag{7}$$

where $\overline{Q}$ — is the main vector of the quantity of motion of the airplane's body and of the liquid in translational motion,

$\overline{K}$ — is the main vector of the moment of the quantities of motion of the airplane's body and of the liquid in translational motion,

$\overline{\omega}$ — is the absolute angular velocity of the coordinate system,

ρ — is the density of the liquid,

$\overline{v}$ — is the relative velocity of the liquid, which is a quantity of the first order of smallness,

$\overline{F}$ — is the main vector of all the external forces,

$\overline{M}$ — is the main moment of all the external forces,

j — is the index of the tank,

τ — is the volume of the liquid.

The equations of relative motion of the liquid (2) in any given tank in it projections on the axes Oxyz will be written in the

form

$$V_0\frac{du}{dt}+\frac{\partial v_{jx}}{\partial t}+\ddot{\theta}z-\ddot{\psi}y=g(-\gamma_0-\theta)-\frac{1}{\rho_j}\frac{\partial p_j}{\partial x}; \qquad (8)$$

$$V_0\dot{\beta}+\frac{\partial v_{jy}}{\partial t}+\ddot{\psi}x-\ddot{\varphi}z=g\varphi-\frac{1}{\rho_j}\frac{\partial p_j}{\partial y}; \qquad (9)$$

$$V_0\dot{\alpha}+\frac{\partial v_{jz}}{\partial t}+\ddot{\varphi}y-\ddot{\theta}x=g-\frac{1}{\rho_j}\frac{\partial p_j}{\partial z}; \qquad (10)$$

$$\frac{\partial v_{jx}}{\partial x}+\frac{\partial v_{jy}}{\partial y}+\frac{\partial v_{jz}}{\partial z}=0, \qquad (11)$$

where ψ is the angle of deflection of the axis from the vertical, a quantity of the first order of smallness.

We will denote the vector of the displacement of the liquid particles in their relative motion by I; then it is possible to write as a first approximation

$$\frac{\partial \bar{l}}{\partial t}=\bar{v}. \qquad (12)$$

We will present the components of the vector of the displacement of the liquid particles in the form

$$l_x=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}(t)\cos\frac{\pi n}{H_y}(y-y_1)\sin\frac{\pi m}{H_x}(x-x_1); \qquad (13)$$

$$l_y=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}b_{mn}(t)\sin\frac{\pi n}{H_y}(y-y_1)\cos\frac{\pi m}{H_x}(x-x_1);$$

$$l_z=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}(t)\cos\frac{\pi n}{H_y}(y-y_1)\cos\frac{\pi m}{H_x}(x-x_1)\left(\frac{z}{z_1}-1\right),$$

where $H_x=x_2-x_1$; $H_y=y_2-y_1$.

If we will substitue these expressions into the equations (7) through (11),

it will be possible to convince ourselves that only those values of the components of the displacement vectors are of interest for which m and n=1, 3,5.........

Due to this fact the expressions (13) simplify to

$$\left.\begin{aligned} l_x &= \sum_{m=1,3,\ldots}^{\infty} a_m(t)\sin\frac{\pi m}{H_x}(x-x_1), \\ l_y &= \sum_{n=1,3,\ldots}^{\infty} b_n(t)\sin\frac{\pi n}{H_y}(y-y_1), \end{aligned}\right\} \tag{14}$$

$$l_z = \sum_{n=1,3,\ldots}^{\infty} c_n(t)\cos\frac{\pi n}{H_y}(y-y_1)\left(\frac{z}{z_1}-1\right)+$$
$$+\sum_{m=1,3,\ldots}^{\infty} c_m(t)\cos\frac{\pi m}{H_x}(x-x_1)\left(\frac{z}{z_1}-1\right).$$

We will integrate (10) in respect to z from the free surface to any given current value of z and by means of the equation so obtained we will eliminate p from (8) and (9). We will substitute into the expression so obtained, and into (11), the components of the displacement vector given by (14); we will multiply them by $\frac{\pi n}{H_y}(y-y_1)$, $\sin\frac{\pi m}{H_x}(x-x_1)$ and by the respective cosines; we will integrate them over the volume /3/; and then we will get the following system of equations:

$$V_0\dot{u}+\ddot{a}_{jm}\frac{\pi m}{4}-\ddot{\psi}\mu_{jy}=g(-\gamma_0-\theta)+\frac{H_{jz}k(\pi m)^2}{4H_{jx}}\ddot{c}_{jm}-$$
$$-\ddot{\theta}\frac{z_{j1}\pi m}{2}+g\frac{k(\pi m)^2}{4H_{jx}}c_{jm}; \tag{15}$$

$$V_0\dot{\beta}+\ddot{b}_{jn}\frac{\pi n}{4}+\ddot{\psi}\mu_{jx}=g\varphi+\frac{H_{jz}k(\pi n)^2}{4H_{jy}}\ddot{c}_{jn}+$$
$$+\ddot{\varphi}\frac{z_{j1}(\pi n)}{2}+g\frac{k(\pi n)^2}{4H_{jy}}c_{jn}; \tag{16}$$

$$a_{jm}+\frac{H_{jx}}{z_{j1}\pi m}c_{jm}=0; \tag{17}$$

$$b_{jn}+\frac{H_{jy}}{z_{j1}\pi m}c_{jn}=0. \tag{18}$$

where

$$H_z = z_2 - z_1; \quad k = \frac{z_2}{z_1}; \quad \mu_x = \frac{x_2 + x_1}{2}; \quad \mu_y = \frac{y_2 + y_1}{2}.$$

The group of equations (15) through (18) will be written separately for each tank.

The substitution of expressions (14) into the equations (7), which were written according to the papers /1/ and /2/, gives the following differential equations:

$$M\frac{du}{dt} + \frac{\rho V_0^2}{2} S\left(c_{x0} + c_{x\alpha}\alpha + c_{x\dot{\alpha}}\frac{\dot{\alpha}l}{2V_0} + c_{xq}\frac{\dot{\theta}l}{2V_0} + \right.$$

$$\left. + c_z\theta + 2\frac{u}{v_0}c_{x0}\right) + Mg\gamma_0 + \sum_{j=1}^{s}\rho_j\frac{2Q_j}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\ddot{a}_{jm}}{m} = 0;$$

$$M(\dot{\beta} + \dot{\psi}) - \frac{\rho V_0 S}{2}\left(c_{y\beta}\beta + c_{y\dot{\beta}}\frac{\dot{\beta}l}{2V_0} + c_{yp}\frac{\dot{\varphi}l}{2V_0} + c_{yr}\frac{\dot{\psi}l}{2V_0}\right) -$$

$$- Mg\frac{\varphi}{V_0} + \sum_{j=1}^{s}\rho_j\frac{2Q_j}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{\ddot{b}_{jn}}{n} = 0;$$

$$M(\dot{\alpha} - \dot{\theta}) + \frac{\rho V_0 S}{2}\left(c_{z0} + c_{z\alpha} + c_{z\dot{\alpha}}\frac{\dot{\alpha}l}{2V_0} + c_{zq}\frac{\dot{\theta}l}{2V_0} + \right.$$

$$\left. + c_{z0}\frac{u}{V_0}\right) - M\frac{g}{V_0}\left(1 - \frac{u}{V_0}\right) = 0;$$

$$I_x\ddot{\varphi} - I_{xz}\ddot{\psi} - \frac{\rho V_0^2 S l}{2}\left(c_{l\beta}\beta + c_{l\dot{\beta}}\frac{\dot{\beta}l}{2V_0} + c_{lp}\frac{\dot{\varphi}l}{2V_0} + \right.$$

$$\left. + c_{lr}\frac{\dot{\psi}l}{2V_0}\right) - \sum_{j=1}^{s}\rho_j\frac{Q_j H_{jy} H_{jz}}{z_{j1}\pi^2}\sum_{n=1,3,\ldots}^{\infty}\frac{\ddot{c}_{jn}}{n^2} -$$

$$- \sum_{j=1}^{s}\rho_j\frac{2Q_j\mu_{jz}}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{\ddot{b}_{jn}}{n} = 0;$$

$$I_y\ddot{\theta} - \frac{\rho V_0^2 S l}{2}\left(c_{m0} + c_{m\alpha}\alpha + c_{m\dot{\alpha}}\frac{\dot{\alpha}l}{2V_0} + c_{mq}\frac{\dot{\theta}l}{2V_0}\right) +$$

$$+ \sum_{j=1}^{s}\rho_j\frac{2Q_j\mu_{jz}}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\ddot{a}_{jm}}{m} + \sum_{j=1}^{s}\rho_j\frac{Q_j H_{jx} H_{jz}}{z_{j1}\pi^2}\sum_{m=1,3,\ldots}^{\infty}\frac{\ddot{c}_{jm}}{m^2} = 0;$$

$$I_z\ddot{\psi} - I_{xz}\ddot{\varphi} - \frac{\rho V_0^2 S l}{2}\left(c_{n\beta}\beta + c_{n\dot{\beta}}\frac{\dot{\beta}l}{2V_0} + \right.$$

$$\left. + c_{np}\frac{\dot{\varphi}l}{2V_0} + c_{nr}\frac{\dot{\psi}l}{2V_0}\right) + \sum_{j=1}^{s}\rho_j\frac{2Q_j\mu_{jx}}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{\ddot{b}_{jn}}{n} -$$

$$- \sum_{j=1}^{s}\rho_j\frac{2Q_j\mu_{jy}}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\ddot{a}_{jm}}{m} = 0,$$

where $Q = H_x H_y H_z$.

The obtained equations represent an infinite system of elementary linear differential equations with constant coefficients.

Their solution is constructed by the reduction method.

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ON ONE METHOD OF FINDING THE ACCELERATION OF A POINT IN THE CASE OF A PLANE-PARALLEL MOTION OF A BODY

By

V. P. Kutler

Let a point M of a flat figure be given an acceleration $\bar{a}_M$. Dividing a_M by the square of the angular velocity of the plane figure we will obtain a vector numerically equal to $\bar{a}_M / \omega^2$ having the direction of $\bar{a}_M$ (fig. 1). We will denote this vector by $\bar{R}_M = \overline{MM}_1$.

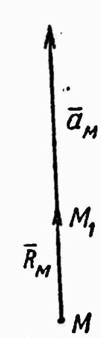

Figure 1

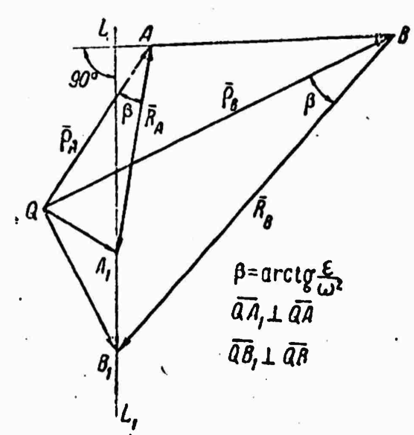

Figure 2

This vector has a linear dimension and is called the reduced acceleration. We will agree that in the further discussion we will denote the reduced acceleration of any point R with an index number identifying the point and on the figure we will denote the tip of the vector by the identifying letter of the point and the index 1.

Theorem: The tips of the reduced accelerations of two arbitrary taken points of the plane figure will lie on the perpendicular

to the straight line connecting these points.

Let Q be the instantaneous center of accelerations of the plane figure, A and B arbitrarily taken points on the plane figure; we now draw the radius vectors ρ_A and ρ_B (fig. 2)

We will express the accelerations of the points A and B by the well known method

$$\bar{a}_A = -\omega^2\bar{\rho}_A + \bar{\varepsilon}\times\bar{\rho}_A; \quad \bar{a}_B = -\omega^2\bar{\rho}_B + \bar{\varepsilon}\times\bar{\rho}_B.$$

Upon dividing by ω^2 we will get

$$\bar{R}_A = \overline{AA_1} = -\bar{\rho}_A + \frac{\bar{\varepsilon}\times\bar{\rho}_A}{\omega^2}; \quad \bar{R}_B = \overline{BB_1} = -\bar{\rho}_B + \frac{\bar{\varepsilon}\times\bar{\rho}_B}{\omega^2}.$$

We will lay off the obtained vectors $\bar{R}_A$ and $\bar{R}_B$, connect the points A_1 and B_1, and will prove that the straight line (we will call it LL_1) on which the points A_1 and B_1 lie is perpendicualr to the straight line AB.

We will draw vectors $\overline{QA}_1$ and $\overline{QB}_1$ from the point Q, from the construction we have $\overline{A_1B_1} = \overline{QB_1} - \overline{QA_1}$. On the other side, also by construction we have

$$\overline{QA_1} = \bar{\rho}_A + \overline{AA_1} = \bar{\rho}_A - \bar{\rho}_A + \frac{\bar{\varepsilon}\times\bar{\rho}_A}{\omega^2};$$

$$\overline{QB_1} = \bar{\rho}_B + \overline{BB_1} = \bar{\rho}_B - \bar{\rho}_B + \frac{\bar{\varepsilon}\times\bar{\rho}_B}{\omega^2}.$$

Consequently

$$\overline{A_1B_1} = \overline{QB_1} - \overline{QA_1} = \frac{\bar{\varepsilon}\times(\bar{\rho}_B - \bar{\rho}_A)}{\omega^2} = \frac{\bar{\varepsilon}\times\overline{AB}}{\omega^2}.$$

As is known $\frac{\bar{\varepsilon}\times\overline{AB}}{\omega^2}$ — is a vector perpendicular to AB, that is the theorem has been proved.

As can be seen, the vector connecting the tips of the reduced acceleration of the two points is numerically equal to the product of the angular velocity by the segment of line that connects these points, divided by the square of the angular velocity of the plane figure

or

$$\left.\begin{aligned} \overline{A_1B_1} &= \frac{\bar{\varepsilon} \times \overline{AB}}{\omega^2} \\ \overline{B_1A_1} &= \frac{\bar{\varepsilon} \times \overline{BA}}{\omega^2}. \end{aligned}\right\} \qquad (1)$$

Consequence of the Above Theorem

The tips of the arrows representing the reduced accelerations of a number of points that lie on the same straight line are located on the perpendicular to the line.

We will use the theorem and result for finding the accelerations of points of the moving plane figure.

Let there be an acceleration $\bar{a}_o$ of the point O whose magnitude and direction is known. Let there also be known the angular velocity ω of the plane figure We will lay off $\bar{R}_0 = \overline{OO_1} = \frac{\bar{a}_0}{\omega^2}$ (fig. 3).

Let the direction of acceleration in another point A of the plane figure

We will denote by AL the straight line on which the vector $\bar{a}_A$ is loacted.

Through the point O_1 we will draw the straight line O_1M perpendicular to the straight line AO to the point of intersection with the

straight line AL. According to the theorem that we have proven, the point of intersection of the straight lines AL and O_1M that is, the point A_1 is the tip of the vector $\bar{R}_A$, the reduced acceleration of the point A.

$$\overline{AA_1} = \overline{R_A} = \overline{AO_1} + \overline{O_1A_1}.$$

Upon multiplying $\bar{R}_A$ by ω^2 we will get $\bar{a}_A = \bar{R}_A\omega^2$. On the basis of equality (1) we have $\overline{O_1A_1} = \frac{\bar{\varepsilon} \times \overline{OA}}{\omega^2}$, from where $\varepsilon = \frac{O_1A_1\omega^2}{OA}$, that is, we find the angular acceleration of the plane figure.

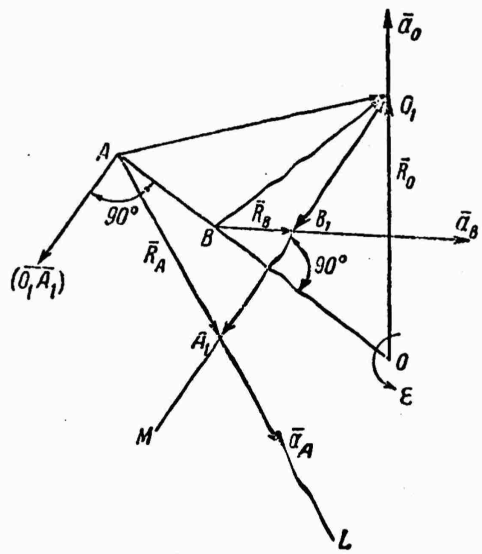

Figure 3

The direction of the vector O_1A_1 obviously coincides with the direction of the tangential acceleration $\bar{a}^{\tau}_{AO}$ of the point A during the rotation of the plane figure around the pole O. By sliding the vector $\overline{O_1A_1}$ to the point A we establish the direction of the curved arrow ε from which it is possible to judge the character of the change of angular velocity ω of the plane figure (increase or decrease). We will also determine the acceleration $\bar{a}_B$ of the point B, situated on the straight line A); the position of this point is considered to be given by $\frac{\overline{OB}}{\overline{OA}} = m$. In accordance with the result of the theorem,

the tip B_1 of the vector $\bar{R}_B$ of the reduced acceleration of the point B is situated on the same straight line $\overline{O_1M}$ which is perpendicular.

We have

$$\bar{R}_B = \overline{BB_1} = \overline{BO_1} + \overline{O_1B_1},$$

in this case

$$\overline{O_1B_1} = \frac{\bar{\varepsilon} \times \overline{OB}}{\omega^2} = \frac{\bar{\varepsilon} \times \overline{OA}}{\omega^2} m = \overline{O_1A_1} m,$$

that is

$$\frac{\overline{O_1B_1}}{\overline{O_1A_1}} = \frac{\overline{OB}}{\overline{OA}} = m.$$

From here it is clear how the vector $\bar{R}_B$ should be constructed; this should be done by dividing the known vector $\overline{O_1A_1}$ in the same proportion m in which the point B divides the vector $\overline{OA}$, and it is necessary to connect the point B_1 to point B. In this manner we will find the vector $\bar{R}_B$ and further

It is possible to find the

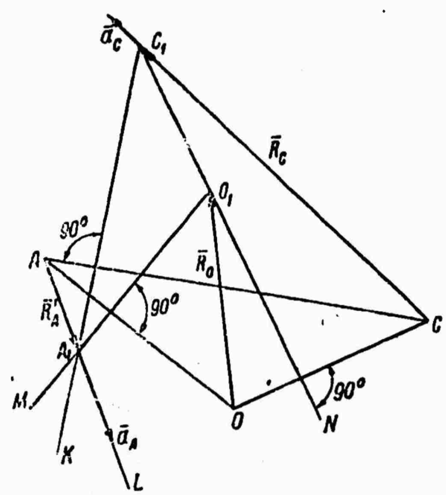

Figure 4

acceleration of any point of the straight line OA by the same method.

Let us now assume that it is desired to find the acceleration of an arbitrary taken point C which does not lie on the straight line OA; in addition, the direction of $\bar{a}_C$ is unknown.

Upon connecting the points O and C (fig. 4) we can claim that the tip C_1 of the vector $\bar{R}_C$ is situated on the straight line O_1N, which passes through the point O_1 perpendicular to OC. We will next connect the point C to the point A. It is obvious that the tip of the vector $\bar{R}_C$ lies on the line A_1K which passes through the point A_1 perpendicular to AC. Consequently, the point C_1 that we wish to find lies at the intersection of the straight lines O_1N and A_1K. Upon connecting the points C and C_1 we will obtain the vector $\bar{R}_C$ and subsequently

As an example we will apply the presented method for finding of acceleration to the solution of the following problem:

Crank OA having a length of 10 cm rotates with an uniform angular velocity $\omega_0 = \sqrt{5}$ sec^{-1} and it imparts its motion to a connecting rod AB having a length of 20 cm. It is required to find the acceleration $\bar{a}B$ of the slider B and the angular acceleration of the connecting rod AB for the position of the mechanism when the angle AOB-30° (fig 5).

Solution: $V_A = \omega_0 OA = 10\sqrt{5}$ cm/sec, the angular velocity of the connecting rod is $\omega = V_A/AP$. From the construction to find $AP = 10\sqrt{5}$ cm, consequently $\omega = 1 sec^{-1}$. We now find the acceleration of the point A:

$$a_A = \omega_0^2 OA = 50 \quad \text{cm/sec}^2$$

Upon determining

$$R_A = AA_1 = \frac{50}{\omega^2} = 50 \text{ см},$$

we will draw through point A_1 the staaight line A_1E perpendicular to AB to the point of intersection with the horizontal line BO along which the acceleration of the point B is directed. B_1, the point of

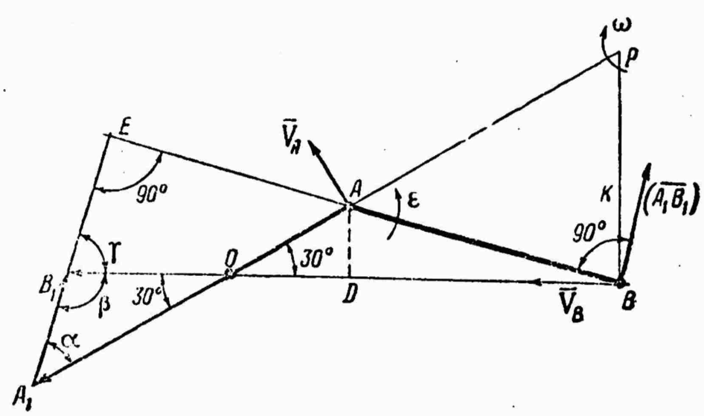

Figure 5
Simple slider-crank mechanism

intersection, is the ip of the reduced acceleration $\bar{R}_B$ of the point B.

For the graphical solution it is convenient to draw the mechanism to the scale 1/5. Upon measuring the section BB_1, using this scale, we will get $\bar{R}_B \approx 57.4$ cm. Consequently $a_B = R_B\omega^2 = 57{,}4$ cm/sec^2. $\bar{R}_B$ can be found by computation.

We have

$$R_B = BB_1 = BD + DO + OB_1; \quad BD = \sqrt{AB^2 - AD^2} = 5\sqrt{15}\ \text{см};$$

$$OD = OA\cos 30^\circ = 8{,}65\ \text{см};$$

$$\frac{OB_1}{\sin\alpha} = \frac{OA_1}{\sin\beta} = \frac{40}{\sin\beta}; \quad \sin\beta = \sin\gamma.$$

We will note that $\widehat{ABD} = AD/AB = 1/4.$

It is possible to assume that the angle ABD approximately equals 15°. Consequently $\gamma = 75°$; $\beta = 105°$; $\alpha = 45°$.

In this manner

$$OB_1 = \frac{40 \sin \alpha}{\sin \beta} = \frac{40 \sin 45°}{\sin 105°} = \frac{20\sqrt{2}}{0{,}96} = 29{,}4 \quad \text{cm}$$

And so

$$R_B = 5\sqrt{15} + 8{,}65 + 29{,}4 = 57{,}4 \text{ см};$$
$$a_B = R_B \omega^2 = 57{,}4 \text{ см/сек}^2.$$

We will now find the angular acceleration of the connecting rod

$$A_1B_1 = \frac{OA_1 \sin 30°}{\sin 75°} = \frac{20}{0{,}96} = \frac{\varepsilon AB}{\omega^2},$$

from where

$$\varepsilon = \frac{20\omega^2}{0{,}96 \cdot 20} = 1{,}04 \quad \text{sec}^2$$

We slide the vector A_1B_1 to the point B keeping the vector parallel to its previous position. We determine the direction of the curved arrow ε. We conclude that the angular velocity of the connecting rod AB diminishes in respect to its absolute magnitude.

Errata

Page	Line	Printed	Should be
60	13th from the bottom	$+\dot{x}xy\ddot{x}+$	$+\dot{x}xy\ddot{x}+$
61	3rd from the top	$+xy\left(\frac{a}{b}-\frac{b}{a}\right)v^2\Delta=0$	$v^2\Delta=0$
63	12th from the top	$+e_{xz}(e_{xy}e_{yz}-e_{yz}e_{xz})$	$+e_{xz}(e_{xy}e_{yz}-e_{yy}e_{xz})$
		$-\frac{\partial(T)}{\partial p_1}\operatorname{ctg}\vartheta p_4-$	$-\frac{\partial(T)}{\partial p_5}\operatorname{ctg}\vartheta p_4-$
148	6th from the bottom	$\frac{\partial(T)}{\partial\dot{\alpha}}=$	$\frac{\partial(T)}{\partial\dot{z}}=$
149	7th from the bottom	$\lvert\dot{\alpha}\dot{\vartheta}\rvert\ll\lvert\sin^3\vartheta_0\rvert\dot{\alpha}^2\ll\lvert\sin^3\vartheta_0\rvert\ \dot{\vartheta}^2\ll\lvert\sin^3\vartheta_0\rvert$	$\lvert\dot{\alpha}\dot{\vartheta}\rvert\ll\lvert\sin^3\vartheta_0\rvert;\ \dot{\alpha}^2\ll\lvert\sin^3\vartheta_0\rvert;$
149	3rd from the bottom		$\dot{\vartheta}^2\ll\lvert\sin^3\vartheta_0\rvert$
		$=\bar{\omega}\times\bar{K}_0$	$=-\bar{\omega}\times\bar{K}_0$
171	4th from the top	$\bar{K}_0'+$	$\dot{\bar{K}}_0+$
175	6th from the bottom	$\bar{\omega}\times[I\bar{\omega}\times\bar{L}]=0$	$\bar{\omega}\times(I\bar{\omega}+\bar{L})=0$
219	11th from the top	$+p\partial(h+$	$+pg(h+$
225	12th from the bottom	$\chi=\frac{l}{R-r}$	$\gamma=\frac{e}{R-r}$
257	6th from the bottom		

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